Monetary Policy Switching and Indeterminacy

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June 18, 2018

Abstract

This paper determines conditions for the existence of a unique rational expectations equilibrium -determinacy- in a monetary policy switching economy. We depart from the existing literature by providing such conditions considering all bounded equilibria. We then apply these conditions to a new Keynesian model with switching Taylor rules. First, deviation from the Taylor principle in one regime does not necessarily cause indeterminacy. Second, very different responses to inflation may trigger indeterminacy even if both regimes satisfy the Taylor principle. Determinacy thus results from the adequacy between monetary regimes rather than the determinacy of each of them taken in isolation.

Keywords: Markov-switching, indeterminacy, monetary policy.

JEL: E31, E43, E52

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1 Introduction

Good monetary policy should prevent indeterminacy, i.e. the existence of multiple stable equilibria. Without a policy tool to coordinate expectations on a particular equilibrium an economy experiencing indeterminacy may respond to non-fundamental -sunspot- disturbances, and hence, may be affected by extrinsic volatility. Since limiting inflation volatility is a widely-accepted objective for monetary policy, extrinsic volatility that is incapable of being controlled is undesirable from a policy perspective. If monetary authorities set the nominal interest rate as a state-contingent rule with constant parameters as suggested by Taylor (1993), preventing indeterminacy requires the nominal interest rate to adjust by more than one-for-one in response to inflation. This condition is known as the Taylor principle.

Monetary policy, however, does not necessarily follow a constant-parameter rule.[1] Many empirical works (for instance Clarida et al., 2000; Lubik and Schorfheide, 2004; Bianchi, 2013) document the existence of monetary policy switching in the post-World War II US economy. As a result economic agents should internalise the possibility of future policy switches when forming their expectations. Since determinacy depends on economic agents' expectations, regime switching affects conditions of stability and therefore requires an update of the Taylor principle.

In a regime switching environment, history-dependent equilibria may emerge making determinacy dependent on restrictions of the class of equilibria. As noted by Farmer et al. (2010a), equilibria of a regime switching model can depend on all past regimes. Previous literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2015) however restricts admissible equilibria by imposing some restrictions in the way equilibria depend on past regimes.

In this paper, we characterise stable equilibria when the economy faces regime switching without restrictive assumptions related to the class of equilibria, and, especially, without excluding equilibria dependent on past regimes. In particular, we study determinacy in the

[1] There are at least three reasons to believe that the parameters of the monetary policy rule may vary over time. First, if monetary policy is optimal, any structural change in the economy should result in a change in monetary policy. Second, the monetary policy rule stems from multiple beliefs regarding the structure of the economy, the role of monetary policy and monetary policy transmission mechanisms. All these beliefs may change over time according to empirical as well as theoretical advances in macroeconomics. Third, the rule captures economic preferences, which have little reason to be stable over time. Governors of major central banks are chosen by the government according to the latter's own preferences, and hence depend on political cycles.

context of a new-Keynesian economy experiencing switching between multiple monetary policy regimes, described as periods for which the interest rate obeys a constant-parameter Taylor rule. Our findings are fourfold.

First, we provide a necessary and sufficient determinacy condition for forward-looking rational expectations models with parameters following a Markov process. This condition requires that the sequence of matrices products dependent on future regimes trajectories converges to a value below one. Yet, in general, this limit cannot be computed analytically as it requires keeping track of an infinite number of trajectories.

Furthermore, we provide tools to apply our theoretical result in commonly used models. First, we extend our results to models with predetermined variables. Second, we provide for an algorithm that checks determinacy in less than 1 second in most of parameters configurations. We establish that the efficiency of the algorithm depends on the norm that is used and we greatly boost up the determinacy checking by choosing an adequate norm.[2]

Second, we settle a controversy in the literature related to conditions of determinacy (Davig and Leeper, 2007; Farmer et al., 2010a; Davig and Leeper, 2010). Using a new-Keynesian model with monetary policy switching, we show that imposing that equilibria depend on a limited number of past regimes leads to an underestimation of the indeterminacy region. All of the existing literature implicitly or explicitly restricts the class of equilibria, see for instance Davig and Leeper (2007), Farmer et al. (2009b), Cho (2015) or Foerster et al. (2016), providing a necessary but not sufficient condition for indeterminacy.[3] To our knowledge, this paper is the first to provide determinacy conditions for the whole class of equilibria.

Third, we apply our results to a two monetary policy regime economy and we prove that the Taylor principle is neither a necessary nor a sufficient condition for determinacy.

Indeed, the Taylor principle is not necessary. One of the regimes can violate the Taylor principle without triggering indeterminacy if monetary policy responds sufficiently (but not too much) to inflation in the other regime. We hence resurrect one of the main findings of Davig and Leeper (2007). Compared to this paper, such policy configurations however appear less often and strong departures from the Taylor principle are not allowed.

[2] All our algorithms are available in an additional separated file.

[3] When a unique stable equilibrium among all possible equilibria exists, the unique stable equilibrium depends only on current shocks and the current regime. Therefore, the unique stable equilibrium always belongs to the classes of equilibria considered in these papers. However, for certain configurations of policy parameters, such restrictions lead to conclude with determinacy while multiple stable equilibria exist.

The Taylor principle is not sufficient either. Indeterminacy can emerge even if both regimes adhere to the Taylor principle. Why does the Taylor principle fail? In a purely forward-looking model, there is always a bounded equilibrium which is the unique equilibrium associated with zero expectations. Indeterminacy thus appears if, and only if, another equilibrium with non-zero expectations exists. As we focus on bounded equilibria, these non-zero expectations have to be consistent with a stable expectations path. For instance in an economy without regime switching, the Taylor principle guarantees that any non-zero expectations will eventually diverge and hence are not admissible. Expectations, however, diverge only asymptotically. In a finite-time horizon, expectations may converge in one direction while diverging in others. These directions may change from one regime to another. Regime switching may thus induce converging expectations.

More concretely, indeterminacy arises when the policy response to inflation changes dramatically from one regime to another. In the monetary regime that reacts the most strongly against inflation, the central banker provokes a large recession to stabilize inflation in case of positive inflation expectations. In finite-time horizon, it means that inflation expectations diverge while output gap expectations converge. In the other regime, the reaction of the central bank to inflation may not ensure that inflation expectations diverge for large output gap expectations. We thus identify cases in which the direction of the convergence of expectations switches from one regime to another. This succession of "incompatible" local behaviors eventually allows for a non-zero stable expectations path and leads to indeterminacy.

Fourth, we show under which conditions the US Great Inflation in the 70s could have been caused by indeterminacy. Clarida et al. (2000) famously suggest that the great volatility in the 70s was the consequence of a violation of the Taylor principle. However, they rely on sub-sample estimations that do not take into account expectations of regime switching. We calibrate a new-Keynesian model following Lubik and Schorfheide (2004) and check determinacy for different transition probabilities. We find that indeterminacy requires a highly persistent violation of the Taylor principle, i.e. the probability of remaining in this regime should be greater than 0.94. Such persistence is consistent with the historical duration of the Great Inflation as well as estimated parameters in the literature.

From a technical side, our stability concept is boundedness and departs from some recent contributions (Farmer et al., 2009b; Cho, 2015; Foerster et al., 2016) which favour the mean square stability concept. We adopt this concept for two main reasons: first, it is the most common concept in the rational expectations literature, second, it is consistent with an underlying non-linear model and a perturbation approach (Barthélemy and Marx, 2017). Of course, the choice of the stability concept matters for determinacy. However, we believe that our main argument -equilibria can depend on past regimes and, following, that determinacy conditions depend on the exact class of equilibria- does not depend on the precise definition of stability. Extending our results to other stability concepts would be a natural avenue for future research.

The remainder of the paper is organised as follows. In section 2, we provide for a simple new-Keynesian model with monetary policy switching that we use throughout the paper to illustrate our results and we depict the restrictions on the solution space, done in the literature, and their consequences. We then turn to a general class of models in section 3. We provide for a necessary and sufficient determinacy condition that we complement with an efficient algorithm to check determinacy in practice. In section 4, we demonstrate that the definition of the solution space is crucial when dealing with regime switching and we provide examples of sunspot equilibria. We illustrate our results with two applications. First, we show the limits of the Taylor principle when monetary policy's reaction to inflation switches between different values, in section 5. Second, we demonstrate that the US Great inflation could only have resulted from a violation from the Taylor principle if economic agents were convinced that this violation was sufficiently long lasting, in section 6. Finally we draw conclusions in section 7.

2 Monetary policy switching

In this section, we present a new Keynesian model with monetary policy switching that we will use throughout the paper and we recall standard determinacy conditions in the absence of regime switching. Then, we review existing findings in the literature and show that the definition of the solution space is critical for determinacy conditions.

2.1 The model

We consider a log-linearised new-Keynesian model following Clarida et al. (2000) and Woodford (2003) in which private decisions satisfy:

$$y_t = \mathbb{E}_t y_{t+1} - \sigma(r_t - \mathbb{E}_t \pi_{t+1} - r_t^n), \tag{1}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + u_t, \tag{2}$$

where variables y_t, π_t and r_t are respectively the output gap, inflation (in log) and the nominal interest rate (in deviation around a certain steady state). The operator $\mathbb{E}_t$ denotes expectations at time t. Equation (1) is an IS curve that links the output gap to all future *ex-ante* real interest rates and future and current shocks, r_t^n. Parameter σ measures risk aversion. Equation (2) is a new-Keynesian Phillips curve linking inflation to all future marginal costs summarized by the output gap. Parameter κ measures the degree of nominal rigidities while β stands for the discount factor. Shock u_t denotes a cost-push shock translating the Phillips curve.

We define a monetary policy regime, denoted by $s_t \in \{1, 2\}$, as a period during which monetary policy abeys a Taylor rule. We suppose that regime s_t follows a Markov proccess characterized by a transition probability matrix P; the probability of switching from regime i to regime j is denoted p_{ij}. The current policy regime is known to private agents while future regimes are not. In regime s_t, the monetary authority sets the nominal interest rate following:

$$r_t = \rho_{s_t} r_{t-1} + (1 - \rho_{s_t})(\alpha_{s_t} \pi_t + \gamma_{s_t} y_t + \varepsilon_t^r), \tag{3}$$

where the parameters α_{s_t} and γ_{s_t} measure the sensitivity of the interest rate to inflation and to the output-gap in each regime. The parameter ρ_{s_t} stands for the inertia in monetary policy decisions. For exposition purposes, we first consider the case $\rho_{s_t} = 0$ and then show in section 3.3 how to extend the results in a more general set up. The shock ε_t^r captures the unsystematic part of monetary policy. Finally, we assume that shocks are bounded[4] and, without loss of generality, are i.i.d. and zero mean.

Then, the question is to determine conditions ensuring the existence of a unique bounded equilibrium.[5] We say that this economy is determinate if it admits a unique bounded equilibrium. Otherwise, it is indeterminate.

By plugging the monetary policy rule, equation (3), into the IS curve, equation (1), we end up with a system of two forward-looking equations simultaneously determining inflation

[4]Boundedness of shocks is required since we are considering bounded equilibria.

[5]As proved below, there is always at least one bounded equilibrium satisfying the model when there is no backward-looking component, ρ_{s_t}. Otherwise, the model may have no bounded equilibrium.

and the output gap:

$$\Gamma_{s_t} z_t = \mathbb{E}_t z_{t+1} + \Omega \varepsilon_t, \tag{4}$$

where the column vector z_t denotes endogenous variables, $[\pi_t \quad y_t]'$ and the column vector ε_t denotes the shocks, $[u_t \quad \varepsilon_t^r \quad r_t^n]'$. The matrices Γ_{s_t} and Ω which gather the parameters of the model are given by

$$\Gamma_{s_t} = \begin{bmatrix} 1/\beta & -\kappa/\beta \\ \sigma(\alpha_{s_t} - 1/\beta) & 1 + \sigma\gamma_{s_t} + \kappa\sigma/\beta \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 1/\beta & 0 & 0 \\ -\sigma/\beta & -\sigma & \sigma \end{bmatrix}.$$

Since shocks are uncorrelated, it seems natural to look for a solution with zero expectations. This is the case for the fundamental equilibrium, denoted by subscript F and defined as $z_t^F = \Gamma_{s_t}^{-1} \Omega \varepsilon_t$, which means:

$$y_t^F = \frac{-\sigma[\alpha_{s_t} u_t - r_t^n + \varepsilon_t^r]}{1 + \sigma\gamma_{s_t} + \sigma\alpha_{s_t}\kappa} \quad \text{and} \quad \pi_t^F = \kappa y_t^F + u_t \tag{5}$$

The question is then whether or not this equilibrium is unique.

2.2 In the absence of regime switching

We start with studying the case without regime switching ($\Gamma_1 = \Gamma_2 = \Gamma$).

The bounded equilibrium (π_t^F, y_t^F) is the only one consistent with zero expectations. It is therefore natural to investigate whether expectations can be different from zero. Following Sims (2002), we thus analyze expectations of inflation and the output gap, which we denote by the column vector $z_t^e = E_t z_{t+1}$. We also introduce the associated forecast error, $\xi_{t+1} = z_{t+1} - z_t^e$. Finally, it is convenient to rewrite equation (4) as follows:

$$z_{t+1}^e = \Gamma z_t^e - \Gamma \xi_{t+1} - \Omega \varepsilon_{t+1}. \tag{6}$$

The equilibrium, z_t^F, is the unique bounded equilibrium if it is impossible to find non-zero stable expectations satisfying equation (6). Suppose that at period 0, expectations are different from zero. Then, depending on the eigenvalues of Γ, expectations will explode or implode. If all eigenvalues of Γ are greater than one then expectations diverge. As we rule out unbounded equilibrium, having non-zero expectations in the first place is inconsistent with a stable equilibrium. This proves that z_t^F is the only stable solution of the model. Thus, determinacy ultimately depends on the lowest eigenvalue of this matrix. This condition is equivalent to: $\alpha + \frac{1-\beta}{\kappa}\gamma > 1$ (see Woodford, 2003).

If monetary authorities do not respond to the output gap, monetary policy can prevent indeterminacy by increasing the nominal interest rate by more than one-for-one in response to inflation. Through such a policy, monetary authorities guarantee that non-zero expectations diverge asymptotically. This result is known as the Taylor principle.

The question we want to address in this paper is how these conditions evolve in the context of a regime switching monetary policy.

2.3 The role of solution space in the literature

Davig and Leeper (2007) provide determinacy conditions for regime switching models assuming that equilibria only depend on the current regime and shocks (see Branch et al., 2007, for a discussion). By denoting by $\mathcal{M}_0$ the set of all bounded equilibria satisfying this property, and by $\mathbf{M} = (P \otimes 1_n) \times diag(\Gamma_1, \Gamma_2)$, Davig and Leeper's result can be restated as follows: there exists a unique bounded equilibrium in $\mathcal{M}_0$ if, and only if, the spectral radius of $\mathbf{M}$, i.e. the largest eigenvalue in absolute value, is strictly less than one. If it is not the case, then any bounded equilibrium in $\mathcal{M}_0$ can be put into the following form:

$$z_t = z_t^F + V_{s_t} w_t \quad \text{and} \quad w_t = J_w w_{t-1} + \xi_t, \tag{7}$$

with, ξ_t being any bounded zero mean process ($\mathbb{E}_t \xi_{t+1} = 0$) independent of current and past regimes. The proof as well as the definition of matrices J_w and V_{s_t} can be found in Appendix A.

Based on this result, Davig and Leeper (2007) prove that one monetary policy regime can fail to satisfy the Taylor principle without endangering the overall determinacy as long as the other regime is sufficiently frequent, long-lasting and that the reaction to inflation in the other regime is sufficiently great. They call this latter result the *long run Taylor principle*.

However, their theoretical results do not hold when considering more general solution spaces and thereby cast doubts about the validity of the long run Taylor principle. Farmer et al. (2010a) have noticed for example that when $\alpha_1 = 3$, $\gamma_1 = 0$, $\alpha_2 = 0.92$, $\gamma_2 = 0$, $p_{11} = 0.8$ and $p_{22} = 0.95$,[6] bounded equilibria exist outside $\mathcal{M}_0$ on top of the fundamental solution. They give the following example:

[6]Other parameters are calibrated as follows: $\beta = 0.99$, $\sigma = 1$ and $\kappa = 0.17$.

$$z_t = z_t^F \texttt{ and } w_t = 0 \texttt{ if } s_t = 1,$$

$$z_t = z_t^F + V w_t \texttt{ and } w_t = \Gamma w_{t-1} + M\xi_t \texttt{ if } s_t = 2,$$

where matrices Γ, V and M are given in Farmer et al. (2010a) and ξ_t is once again an i.i.d. zero-mean shock.

The reason why the Davig and Leeper (2007) result does not rule out this additional bounded equilibrium is that the latter depends on past regimes and therefore does not belong to $\mathcal{M}_0$. The additional part w_t of the above equilibrium depends on past w_{t-1} which implicitly depends on *all* past regimes, while in equation (7), this additional part does not depend on past regimes.

In fact Farmer et al. (2010b) prove that any equilibrium can be written as follows:

$$z_t = z_t^F + V_{s_t} w_t, \tag{8}$$

$$w_t = \phi_{s_{t-1}s_t} w_{t-1} + V'_{s_t}\xi_t, \tag{9}$$

where V_{s_t} and $\phi_{s_{t-1}s_t}$ are regime dependent matrices and ξ_t is an arbitrary zero mean process.

The sunspot component ξ_t may depend on past regimes in a sophisticated way, such that checking the stability of the process w_t is hard. To circumvent this issue, the literature (Farmer et al., 2010b; Cho, 2015, for instance) supposes that the sunspot part ξ_t does not depend on past shocks or regimes. There is, however, no clearcut reason to make such an assumption. Indeed, the process ξ_t may depend on past regimes and we show in section 4 that its structure affects the stability of the process w_t. In next section, we establish the determinacy conditions which are valid whatever the structure of the process ξ_t and therefore do not depend on a particular restriction to the solution space.

3 Determinacy conditions

In this section, we first present the class of models we deal with (section 3.1). Second, we derive determinacy conditions for purely forward-looking regime switching models without assuming restrictions on the solution space (section 3.2). We then extend our findings to regime switching models with backward-looking components (section 3.3). Finally, we provide a concrete and simple application and describe algorithms that check our determinacy conditions efficiently (section 3.4).

3.1 The class of models

Most micro-founded macroeconomic models may be summarised by a system of non-linear equations involving structural parameters governing economic agents' preferences, technology, market structures and economic policies. Allowing these parameters to switch over time results in non-linear regime-switching models. When shocks are small enough, the stability of this class of models can be checked by studying the determinacy of linear regime-switching models as demonstrated by Barthélemy and Marx (2017). In this paper, we focus on linear models of the following form:

$$A_{s_t}\mathbb{E}_t z_{t+1} + B_{s_t} z_t + C_{s_t} z_{t-1} + D_{s_t}\varepsilon_t = 0, \tag{10}$$

where index t denotes time and belongs to $\{-\infty, \dots, \infty\}$, vector z_t is a $(n \times 1)$ real vector of endogenous variables, vector ε_t is a $(p \times 1)$ real vector of exogenous shocks and index s_t indicates the current regime, in $\{1, \cdots, N\}$. For any index $i \in \{1, \cdots, N\}$, matrices A_i, B_i and C_i are $(n \times n)$ real matrices. We assume that matrices A_i and B_i are invertible. The vector of shocks ε_t is assumed to be i.i.d. and bounded. Finally, we assume that regimes follow a Markov-chain with constant transition probabilities:

$$\forall (i,j) \in \{1, \cdots, N\}^2,\ Pr(s_t = j | s_{t-1} = i) = p_{ij}, \tag{11}$$

where the scalar p_{ij} is between 0 and 1 and naturally sums to 1 over regimes j. We thus assume that transition probabilities are constant over time and over state.

We consider that an equilibrium is stable if it is bounded. Precisely, we assume that F is a bounded set such that $\varepsilon_t \in F$, and we denote by $\varepsilon^t = \{\varepsilon_t, \cdots, \varepsilon_{-\infty}\}$ and $s^t = \{s_t, \cdots, s_{-\infty}\}$ the history of shocks and regimes. We define a stable equilibrium as follows:

Definition 1. *A stable equilibrium is a function z on $\{1, \cdots, N\}^\infty \times F^\infty$, satisfying the model* (10) *and such that*

$$\|z\|_\infty = \sup_{z^t, \varepsilon^t} \|z(s^t, \varepsilon^t)\| < \infty. \tag{12}$$

We denote by $\mathcal{B}$ the set of all the bounded functions on $\{1, \cdots, N\}^\infty \times F^\infty$. The set $\mathcal{B}$, with the norm $\| \cdot \|_\infty$ defined in equation (12) is a Banach space.

Following the literature (Blanchard and Kahn, 1980; Lubik and Schorfheide, 2004), we say that the model/the economy is determinate if there exists a unique stable equilibrium. For purely forward-looking models ($C_{s_t} = 0$), when the model is not determinate there are multiple stable equilibria. Otherwise, when the model is not determinate there are two distinct cases: either there are multiple stable equilibria or no stable equilibrium.

This definition leads to two remarks.

First, the stability concept in Definition 1 is similar to the one used in non-linear rational expectations models (e.g. Woodford, 1986; Jin and Judd, 2002). Farmer et al. (2009b) and many others derive determinacy conditions for mean square stable equilibria belonging to a subclass of equilibria that we describe in Section 4. We choose to stick to boundedness for the reasons described in the introduction.

Second, we follow closely Woodford (1986) by assuming that time begins at $-\infty$, and hence that there is no initial condition. We thus search for *stationary* stochastic processes that map all the past and current shocks and regimes to current endogenous variables. Such a definition is convenient because it allows us to apply usual techniques to study the spectrum of isometries in Banach spaces (asin Conway, 1990). Absent regime switching, whether the starting date is finite or not is not crucial.[7] In section 4, we discuss how to reinterpret indeterminacy in a regime switching environment without an initial date, as a problem of multiple initial conditions in an environment with an initial date.

From an economic perspective, assuming that the model is valid from $t = -\infty$ onwards, ensure that we do not underestimate the indeterminacy region by arbitrary restricting the dependency of equilibria of past regimes. The related drawback is that we may consider *exotic* equilibria that some readers may find unrealistic. The definition of what is *exotic* and *realistic* being subjective and dependent of the considered problem, we prefer adopting an agnostic view in this paper.

3.2 Determinacy conditions for purely forward-looking models

This subsection establishes the necessary and sufficient conditions for the existence and uniqueness of a stable equilibrium when the model is purely forward-looking, i.e. when the

[7] Proposition 4 shows that determinacy conditions do not depend on this specific assumption.

matrices $C_{s_t} = 0$ for all $s_t \in \{1, \dots, N\}$. The model can then be rewritten as follows:

$$\Gamma_{s_t} z_t = \mathbb{E}_t z_{t+1} + \Omega_{s_t} \varepsilon_t. \tag{13}$$

where matrices $\Gamma_{s_t} = -A_{s_t}^{-1} B_{s_t}$ and $\Omega_{s_t} = A_{s_t}^{-1} D_{s_t}$. Notice that matrices Γ_i are invertible since matrices A_i and B_i are supposed to be invertible.

Let us first observe that equation (13) can be rewritten in terms of expectations as follows:

$$z_t^e = \Gamma_{s_t} z_{t-1}^e - \Gamma_{s_t} \xi_t - \Omega_{s_t} \varepsilon_t, \tag{14}$$

where the vector z_t^e is the expectation at time t of endogenous variables at time $t+1$, i.e. $\mathbb{E}_t z_{t+1}$ and the column vector ξ_t is the associated forecast errors, $z_t - \mathbb{E}_{t-1} z_t$. Determinacy conditions of the original model are equivalently given by the existence of a unique couple (z^e, ξ) of bounded process and zero mean process satisfying equation (14). By pre-multiplying this equation by $\Gamma_{s_t}^{-1}$ and taking the expectations, we get the following relation:

$$\mathbb{E}_t \Gamma_{s_{t+1}}^{-1} z_{t+1}^e = z_t^e, \tag{15}$$

which leads to, for any integer $k > 1$,

$$z_t^e = \mathbb{E}_t (\Gamma_{s_{t+1}}^{-1} \cdots \Gamma_{s_{t+k}}^{-1}) z_{t+k}^e. \tag{16}$$

As in subsection 2.2, we see that the model is determinate if and only if the only stable solution of (16) is zero. This relies on a sufficient decrease of $\mathbb{E}_t(||\Gamma_{s_t}^{-1} \Gamma_{s_{t+1}}^{-1} \cdots \Gamma_{s_{t+k}}^{-1}||)$. We denote by the scalar u_k, a sequence that measures the rate of decrease of the expected products of Γ_i:

$$u_k = \left(\sum_{(i_1, \cdots, i_k) \in \{1, \cdots, N\}^k} p_{i_1 i_2} \cdots p_{i_{k-1} k} \| \Gamma_{i_1}^{-1} \Gamma_{i_2}^{-1} \cdots \Gamma_{i_k}^{-1} \| \right)^{1/k}. \tag{17}$$

The determinacy condition is then given by:

Proposition 1. *There exists a unique stable equilibrium if and only if the limit of u_k, when k tends to infinity, is smaller than 1. We denote by ν such a limit. When unique, this equilibrium is then given by $z_t = \Gamma_{s_t}^{-1} \Omega_{s_t} \varepsilon_t$.*

We prove Proposition 1 in Appendix, section B. We first prove that the sequence u_k converges and admits a limit independent of the chosen norm. If the limit is smaller than 1, then the sequence $\mathbb{E}_t\Gamma^{-1}_{s_{t+1}}\cdots\Gamma^{-1}_{s_{t+k}}X_{t+k}$ tends to 0 whatever the bounded stochastic process X and hence the model admits a unique stable equilibrium. We prove the reciprocal by showing that if the limit is greater than one we can construct multiple bounded equilibria, by applying the Gelfand theorem. Finally, when unique, the unique equilibrium depends only on the current regime and shocks (see appendix A, equation (3)).

Proposition 1 extends standard determinacy conditions to Markov-switching rational expectations models. When there is no regime switching ($\Gamma_{s_t} = \Gamma$ for any regime), the existence of a unique stable equilibrium depends on the asymptotic behavior of $u_k \sim ||\Gamma^{-k}||^{1/k}$. This sequence behaves as a decreasing exponential if and only if all the eigenvalues of Γ^{-1} are less than one. This coincides with the well-known Blanchard and Kahn conditions.

This Proposition also extends the Farmer et al. (2009a) results to multivariate models. When the model is univariate, the matrices Γ_{s_t} are commutative as they are scalars. The limit of u_k therefore depends on a simple combination between probabilities and these scalars.

However, in general, the computation of the limit is challenging as the number of terms to compute grows exponentially. This complexity comes from the non-commutativity of the products of the matrices appearing in the definition of the sequence, u_k. Consequently, the speed of convergence is unknown.[8] Behind this complexity it may be intuited that under some circumstances, economic agents' decisions may depend on the exact order of future regimes and the related expectations conditional on the considered path. Consequently, shocks and regimes in the distant past may impact current decisions in the case of indeterminacy. The complexity of finding determinacy conditions has been already mentioned by Costa et al. (2005) and Farmer et al. (2009b).

Proposition 1 illustrates that the determinacy condition in general is highly complex, conceptually and numerically. Proposition 2 proves that it is not necessary to compute higher order terms of the sequence u_k as long as u_k is lower than one. Indeed, if one term of the sequence is lower than one, it is sufficient to prove determinacy.

Proposition 2. *If there exists k such that $u_k < 1$, then there exists a unique equilibrium.*

The proof of Proposition 2 is in Appendix, section D. If it can be proven that there exists

[8] The limit of sequence u_k shares similar properties with mathematical objects such as the joint spectral radius (e.g. Theys, 2005) and the p-radius. For instance, for a two-regime model, if the transition probabilities are symmetric ($p_{11} = p_{22} = 1/2$), the limit of u_k when k tends to infinity exactly corresponds to the 1-radius of $\{\Gamma_1^{-1}, \Gamma_2^{-1}\}$. We refer to Jungers and Protasov (2011) for a detailed presentation of this quantity. The complexity of these concepts is therefore well-known in control theory.

k such that u_k is smaller than 1, then its limit is smaller than 1. This condition converges to the determinacy frontier established in Proposition 1 as k tends to infinity.

The reasoning is as follows. Assume that u_k is lower than one. The question is whether non-zero expectations, z_t^e, are consistent with a bounded equilibrium satisfying equation (16). Suppose that these expectations are non-zero. Because $u_k < 1$, we see that expectations increase every k periods, i.e. $||z_t^e||/(u_k^p) < ||z_{t+kp}||$. Expectations thus diverge. This proves that $z_t^e = 0$ is the only possible equilibrium consistent with bounded expectations. This reasoning is similar to the standard Blanchard and Kahn (1980) forward iteration but instead of studying the relationship between the current and immediate future periods, we link the current economic outcome with economic agents' expectations k-period ahead.

3.3 Determinacy conditions for general class of models

In this subsection, we extend previous results to solve general models under the form (10), and we derive determinacy conditions for this extended class of models by combining our previous results and an approach developed by Cho (2015). This latter paper links the determinacy conditions of a purely forward-looking model to those of the model with predetermined variables. This method is also close to the undetermined coefficient approach detailed in Uhlig (1999). Our method relies on three main steps.

The first step is to find invertible matrices $\{R_1, \cdots, R_N\}$ solving the system of matricial equations:

$$R_{s_t} = B_{s_t} - \mathbb{E}_t A_{s_t} R_{s_{t+1}}^{-1} C_{s_{t+1}}, \tag{18}$$

such that the joint spectral radius of $\{R_1^{-1}C_1, ..., R_N^{-1}C_N\}$ is smaller than one.[9] We denote by $\rho(\{R\})$ such a joint spectral radius. Cho (2015) and Foerster et al. (2016) have recently provided algorithms solving this system of matricial equations (see also Maih, 2015).

The second step consists in defining a new variable w_t as a linear combination of an arbitrary bounded solution of the model (10) z_t:

$$w_t = z_t - R_{s_t}^{-1} C_{s_t} z_{t-1}. \tag{19}$$

[9] The joint spectral radius is the maximum growth rate of the product of matrices. For a set of matrices $\{M_1, ..., M_N\}$, the joint spectral radius is defined as $\rho(M_1, ..., M_N) = \limsup_k \{||M_{s_1} ... M_{s_k}||^{1/k}\}$.

Notice that, once R_{s_t} is fixed, there are as many bounded solutions to model (10) as variables w_t. In addition, nothing guarantees that the newly created variable w_t behaves nicely (it can be non-Markovian if z_t is non-Markovian for instance). Since the variable z_t is a solution to model (10), the new variable w_t must satisfy the following purely forward-looking model:

$$A_{s_t}\mathbb{E}_t w_{t+1} + R_{s_t} w_t + D_{s_t}\varepsilon_t = 0. \tag{20}$$

The third step is to check whether equation (20) admits a unique solution. If yes, then there exists also a unique solution to the initial model (10). The determinacy conditions of the forward-looking model (20) will thus provide the determinacy conditions of the initial model (10). According to Proposition 1, these determinacy conditions will depend on the limit of a sequence v_k denoted by $\nu(\{R\})$ and defined, as in section 3.2, by:

$$\nu(\{R\}) = \lim_{k\to\infty}\left(\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}k}\|R_{i_1}^{-1}A_{i_1}R_{i_2}^{-1}A_{i_2}\cdots R_{i_k}^{-1}A_{i_k}\|\right)^{1/k}.$$

We index by the superscript $h \in \{1,\ldots,H\}$ the different sets of matrices $\{R^h\} = \{R_1^h,\ldots,R_N^h\}$ that are solutions to the matricial equations (18). Based on the analysis of the associated joint spectral radius $\rho(\{R^h\})$ and limit $\nu(\{R^h\})$, Proposition 3 allows to check determinacy in most circumstances.

Proposition 3.

1. *If there exist h such that $\rho(\{R^h\}) < 1$ and $\nu(\{R^h\}) < 1$, then the model admits a unique stable equilibrium given by:*

$$z_t = (R_{s_t}^h)^{-1}C_{s_t}z_{t-1} - (R_{s_t}^h)^{-1}D_{s_t}\varepsilon_t; \tag{21}$$

2. *If there exist h such that $\rho(\{R^h\}) < 1$ and $\nu(\{R^h\}) > 1$, then the model admits multiple stable equilibria;*

3. *If there exist h such that $\rho(\{R^h\}) > 1$ and $\nu(\{R^h\}) < 1$, then the model admits no stable equilibrium;*

4. *If for all h, $\rho(\{R^h\}) > 1$ and $\nu(\{R^h\}) > 1$, then we cannot conclude.*

Proof. The proof of Proposition 3 is given in Appendix, section E. The structure of the model is such that the case 3 is disconnected from cases 1 and 2. The first case ensures that there exists a set of matrices $\{R_1^h, \cdots, R_N^h\}$ satisfying equation (18) whose joint spectral radius $\rho(\{R^h\})$ is less than one, and thus that there exists at least one stable equilibrium defined by equation (21). Then Proposition 1 proves that this equilibrium is unique if and only if the limit $\nu(\{R^h\})$ is less than one. The second case is an extension of Proposition 1. The third case ensures that there only exists unbounded equilibria. In the fourth case, we cannot conclude. □

This proposition is forceful since it links the determinacy of models with predetermined variables with the determinacy of purely forward-looking models like those described by equation (13). The stability condition on the spectral radius $\rho(\{R^h\})$ ensures that, for a given bounded process w_t, the process $z_t = (R_{s_t}^h)^{-1} C_{s_t} z_{t-1} + w_t$ is a bounded process while Proposition 3 ensures that there exists a unique bounded process w_t.

Let explain how to use Proposition 3 in practice. First, find a particular solution $\{R^h\}$ to the system of quadratic equations (18). Second, compute the limit $\nu(\{R^h\})$ and the spectral radius $\rho(\{R^h\})$ and check whether one of the first three items of Proposition 3 is satisfied. If yes, then determinacy is settled. Otherwise, find another solution to equation (18), compute the limit, the spectral radius and so on. If after many trials, all the solutions fit item 4 of Proposition 3, our method is inconclusive. However, we have never found such a dead end up to now; in the absence of regime switching, this cannot occur (see item 4, Proposition 4). In subsection 3.4, we apply this Proposition to a simple example that can be solved analytically, we also provide for a detailed algorithm (see Algorithm 2) to apply this Proposition in practice, and we give methods to boost up the speed of the computation of the limit $\nu(\{R^h\})$.

Without regime switching Let us show that Proposition 3 coincides with the standard Blanchard and Kahn conditions absent of regime switching. Blanchard and Kahn (1980) proves that the characterization of equilibria in the absence of regime switching relies on the number of explosive eigenvalues ($n_F \in \{0, \ldots, 2n\}$) of the matrix F defines as follows:

$$F = \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ \mathbb{1} & 0 \end{bmatrix},$$

where we temporarily omit the indexes s_t and denote by A, B, C and D the matrices of model (10), the model is then rewritten as:

$$A\mathbb{E}_t z_{t+1} + Bz_t + Cz_{t-1} + D\varepsilon_t = 0. \tag{22}$$

Let denote by $\{R^h\}$ with $h \in \{1, \cdots, H\}$ the H matrices that satisfy the matricial equation (18):

$$R = B - AR^{-1}C. \tag{23}$$

We denote by $\rho(X)$ the spectral radius of matrix X as a special case of the joint spectral radius of a set of matrices. Proposition 3 can be rewritten as follows:

Proposition 4. *(Blanchard and Kahn, 1980)*

1. *If there exists $h \in \{1, \cdots, H\}$ such that $\rho((R^h)^{-1}C) < 1$ and $\rho((R^h)^{-1}A) < 1$, then $n_F = n$ and the model admits a unique stable equilibrium;*
2. *If there exists $h \in \{1, \cdots, H\}$ such that $\rho((R^h)^{-1}C) < 1$ and $\rho(A(R^h)^{-1}) > 1$, then $n_F > n$ and the model admits multiple stable equilibria;*
3. *If there exists $h \in \{1, \cdots, H\}$ such that that $\rho((R^h)^{-1}C) > 1$ and $\rho(A(R^h)^{-1}) < 1$, then $n_F < n$ and the model admits no stable equilibria;*
4. *The case where, for any $h \in \{1, \cdots, H\}$, $\rho((R^h)^{-1}C) > 1$ and $\rho(A(R^h)^{-1}) > 1$ is empty.*

Proof. The proof combines known results on quadratic equations. The structure of model (22) is such that cases 1 and 2 are disconnected (which is not obvious a priori). All the details are in Appendix, section F □

Proposition 4 gives an alternative characterization of equilibria based on matrices R defined in equation (18). This characterization is less elegant than the one based on eigenvalues of the matrix F but formally proves that our method is equivalent to Blanchard and Kahn (1980) conditions in the absence of Markov switching.

3.4 Determinacy conditions in practice

In this subsection, we explain how to manipulate determinacy conditions derived above practically. We first show how to apply Proposition 3 in a flexible prices model with a switching

Taylor rule responding to past inflation. Because the model is univariate, we can easily compute all the objects needed to apply Proposition 3 (especially the sequences of matrices R^h). We then provide for an algorithm that reduces the time needed to check determinacy conditions of purely forward-looking models when an analytical resolution is impossible. We show the time gains in the New Keynesian model in which checking determinacy conditions using brute force may be very time consuming in certain parameters regions. We finally describe an algorithm to check determinacy conditions for the general class of models including backward-looking components.

A simple application of Proposition 3 We modify the New-Keynesian model along two lines: (i) prices are flexible (the parameter κ tends to $+\infty$) and hence the output gap $y_t = 0$ for all t (ii) the nominal interest rate responds to past inflation:

$$i_t = \alpha_{s_t}\pi_t + \mu_{s_t}\pi_{t-1} + r_t^n, \tag{24}$$

where the positive scalar μ_{s_t} measures the response of the nominal interest rate to past changes in inflation. Equations (24) and (1) lead to the simple univariate equation:

$$\mathbb{E}_t\pi_{t+1} = \alpha_{s_t}\pi_t + \mu_{s_t}\pi_{t-1}.$$

This model fits well Proposition 3 with matrices $A_1 = A_2 = 1$, $B_1 = -\alpha_1$, $B_2 = -\alpha_2$, $C_1 = -\mu_1$ and $C_2 = -\mu_2$. Without loss of generality we assume that the economy never remains in state 2 ($p_{22} = 0$). The matrices $\{R^h\}$ in proposition 3 are scalars r_1 and r_2, solutions to the two quadratic equations:

$$[p_{11}r_1 + (1-p_{11})r_2]\, r_1 - \alpha_1 r_1 - \mu_1 = 0,$$
$$r_1 r_2 = \alpha_2 r_2 + \mu_2.$$

Depending on the roots of these equations, Proposition 3 characterizes the equilibria. Table 1 illustrate how to concretely apply Proposition 3 for four values of the pair (α_1, α_2). We calibrate other parameters to $\mu_1 = 0.2$, $\mu_2 = 1.2$, $p_{11} = 0.5$. The first column reports the value of the pair (α_1, α_2); the second column reports all the roots of the polynomial described above (r_1, r_2); the third column and the fourth column report the computation of the joint spectral radius $\rho(\{R\})$ and the limit $\nu(\{R\})$ for each root; finally, the last column concludes based on Proposition 3. Notice that because the model is univariate, we can easily compute the joint spectral radius as well as the limit. Figure 7 in Appendix reports the different regions

(α_1, α_2)	(r_1, r_2)	$\rho(\{R\})$	$\nu(\{R\})$	Case of Proposition 3
	$(1.6 - 0.4i, -1.2 + 0.5i)$	0.9	0.7	1
$(0.2, 2.5)$	$(1.6 + 0.4i, -1.2 - 0.5i)$	0.9	0.7	
	$(-2.1, -0.3)$	4.6	2.2	
	$(-0.1, -1.1)$	1.9	5.2	
$(1, 1)$	$(1.6 - 1.1i, 0.4 + 0.9i)$	1.2	0.8	3
	$(1.6 + 1.1i, 0.4 - 0.9i)$	1.2	0.8	
	$(1.5, 2.6)$	0.5	0.5	1
$(2, 1)$	$(-0.1, -1.1)$	3.3	8.8	
	$(4.4, 0.4)$	3.4	1.5	
	$(-0.0, -6.0)$	195.5	488.7	
$(2.5, 0.2)$	$(0.5, 4.6)$	0.4	1.2	2
	$(0.5, 4.6)$	6.3	2.7	

Table 1: Illustration of proposition 3 for different policy parameters (α_1, α_2)

Note: See the text for detailed explanation of each column. Each group of lines correspond to a pair of policy parameter (α_1, α_2). We surround the walues of $\rho(\{R\})$ and $\nu(\{R\})$ allowing to conclude in which case of Proposition 3 we are. Case 1 corresponds to a region of parameters that is consistent with a unique equilibrium (determinacy), Case 2 with multiple equilibria (indeterminacy), and Case 3 with no equilibria. In this univariate example $\rho(\{R\}) = \max\{\frac{|\mu_1|}{|r_1|}, \frac{|\mu_2|}{|r_2|}\}$ and $\nu(\{R\}) = \frac{1-p_{11}}{|r_1|} + \frac{p_{11}}{|r_2|}$.

of determinacy, indeterminacy and no-equilibria with respect to the pair (α_1, α_2). This graph shows that we never find the fourth case of Proposition 3 in this simple example.

Boosting the convergence process Except very special cases such as the univariate environment presented above, computing the limit involved in Proposition 1 may be time-consuming. The speed of convergence of the sequence u_k is sensitive to the choice of the matricial norm $\|.\|$. Though there is no known method to find the best matricial norm, we propose a choice of norm that proves to be efficient in most applications. From any matricial norm, $\|.\|$, we can derive many norms by changing the basis of the vectorial space. We denote by $\|.\|_Q$ such a norm:

$$\|A\|_Q = \|Q^{-1}AQ\|, \tag{25}$$

where Q is an arbitrary invertible matrix. Let define Q_k^* the optimal matrix minimizing u_k for an arbitrary $k > 0$ with respect to matrix Q. We then define the associated sequence $u_k^\star$ as follows:

$$u_k^\star = \left(\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}k} \|\Gamma_{i_1}^{-1}\Gamma_{i_2}^{-1}\cdots\Gamma_{i_k}^{-1}\|_{Q_k^*} \right)^{1/k}.$$

We now prove that this new sequence converges toward the same limit as the initial sequence u_k and can thus be used in practice to check determinacy of Markov switching rational expectations models.

Proposition 5. *The sequence* (u_k^*) *and the sequence* (u_k) *admit the same limits.*

Proof. Let us consider the sequence u_k^* defined when considering the matricial norm $\|.\|_{Q_k^*}$ for all k. It is evident in view of Proposition 2, that this sequence is above the limit. Furthermore, this sequence is always smaller than u_k (the sequence associated to $Q = 1_n$), which converges to ν according to Proposition 1. As a consequence, u_k also converges to ν. □

Proposition 5 gives the best norm among the subset of norms generated by a basis change in the vectorial space.

While this optimization is potentially costly for large k, it is really powerful in enhancing the accuracy of the approximation of the limit ν. Thus we recommend finding first the best matricial norm for small k (let say $k = 5$) in order to find a matrix that is convenient for the considered problem and then iterate on k without changing the matricial norm anymore. This method is both accurate and fast in most applications.

Let us now show the practicality of Proposition 5 in the monetary policy switching model presented in Section 2 when the Taylor rule does not incorporate backward-looking component ($\rho_{s_t} = 0$). In figure 1, we plot the determinacy region for parameters calibrated as in Davig and Leeper (2007). The determinacy region is significantly smaller than in their paper explaining why Farmer et al. (2010a) can find policy parameters satisfying their determinacy conditions but consistent with multiple bounded equilibria. We discuss economic consequences of this determinacy frontier in sections 5 and 6.

The choice of the norm appears critical for the accuracy of the approximation of the limit. Figure 2 reports the values of u_k for different choices of norms with respect to the computing

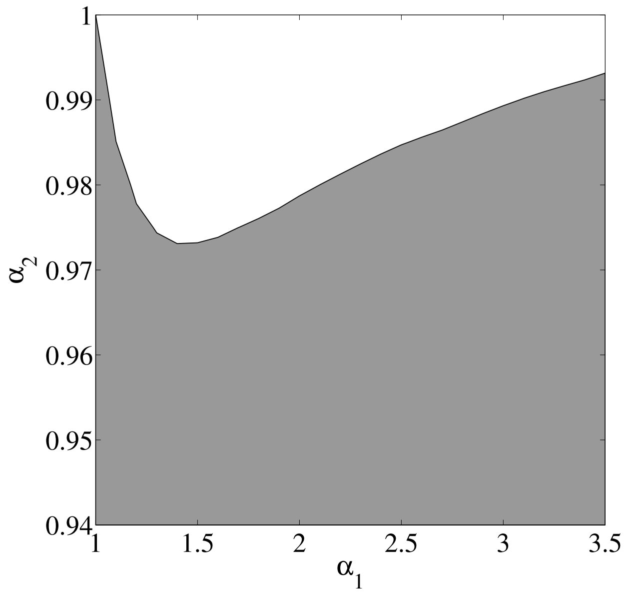

Figure 1: Determinacy condition and policy parameters

Note: the white area depicts the determinacy region with respect to policy parameters α_1 *and* α_2*. Other parameters are calibrated as follows:* $\gamma_1 = 0$*,* $p_{11} = 0.8$*,* $\gamma_2 = 0$*,* $p_{22} = 0.95$*,* $\kappa = 0.17$*,* $\beta = 0.99$ *and* $\sigma = 1$*. The approximation of the limit* ν *is computed by computing* u_{20} *with the optimized norm* $\|.\|_{Q^*_{15}}$*.*

time (in log scaled). The black line with diamonds plots the sequence when we choose the optimal transition matrix Q^*_k for all k. Compared to other norms, this is the most precise. However, the minimization is computationally costly[10]. In practice we recommend computing first the optimized norm, $||.||_{Q^*_5}$ and then iterating to compute the sequence of u_k. This choice leads to an approximated value of the limit similar to the optimal matrix choice but without implying such a large computational cost. In the calibration used for the figure, determinacy can be settled in less than a second. We thus propose the following algorithm:

Algorithm 1. *Determinacy conditions for purely forward-looking models*

1. *Choose a norm (among norms induced by a vectorial norm)*
2. *Compute the optimal basis change matrix* Q^*_5 *(this can be done by using exponential matrices as a basis of the unitary matrix)*
3. *Compute* u_k *for the matricial norm* $\|.\|_{Q^*_k}$ *for* $k > 5$

[10]This minimization can be computed analytically in some circumstances (depending on the chosen norm). But the gain to compute it numerically appears small as the analytical result is difficult to manipulate.

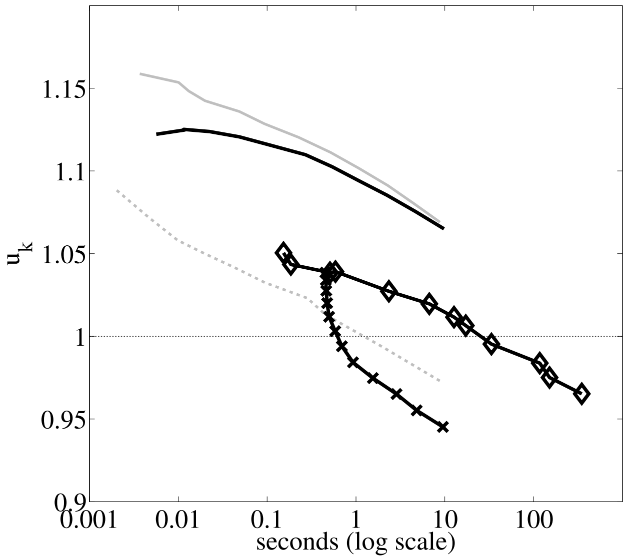

Figure 2: Speed of convergence, accuracy and norm choice

Note: the y-axis represents u_k for different chosen matricial norms for $k < 16$. Thex-axis reports time in seconds (log-10 axis). The black line reports matricial norm induced by the vector 2-norm. The grey plain line with crosses (dotted line, plain line) reports convergence for norms induced by the vector ∞-norm, induced by the vector 1-norm and the standard Frobenius norm respectively. Along the black line with crosses, we plot the sequence u_k when we choose the basis change matrix Q_5^*, i.e. optimized for u_5, and along the black line with diamonds, when we choose the optimal matrix Q_k^* for all k as in Proposition 5. Policy parameters are calibrated as follows: in regime 1, $\alpha_1 = 1.4$, $\gamma_1 = 0$, and $p_{11} = 0.8$ while $\alpha_2 = 0.97$, $\gamma_1 = 0$, and $p_{11} = 0.95$. Programs are launched on Intel(R) Core(TM) i7-4770 CPU 3.4GHz machine using Matlab R2014a.

4. *Stop if u_k is lower than one or if the increment from k to $k+1$ is negligible compared to the distance to 1 otherwise iterate step 3 for a larger k.*

5. *If u_k<1 the model is determinate, indeterminate otherwise.*

This algorithm is available online and describes the computations required to get the black line with crosses in Figure 2. In most cases, we found that the norm induced by the vector 1-norm is the one providing the best speed/accuracy trade-off; however, we cannot reject that for some economic problems, other norms should be preferred.

General algorithm Let us now present our proposed algorithm to solve and check determinacy of regime switching models when introducing backward-looking components. We closely follow the three steps identified in section 3.3. The first step is to find a set of matrices solving (18); the second step consists in checking the stability of this set; the last step is to check the determinacy conditions of the forward-looking model associated with this set of matrices.

The algorithm is as follows:

Algorithm 2. *Determinacy conditions for general models*

1. *Find matrices* $\{R_{s_t}\}$ *satisfying equation* (18)*. To do so, one can use the forward iteration procedure developed by Cho (2015).*

2. *Compute the joint spectral radius of this set of matrices,* $\rho(\{R\})$*. We use the program developed by Vankeerberghen et al. (2014) which provides upper and lower bounds of the joint spectral radius efficiently.*

3. *Compute the limit* $\nu(\{R\})$ *using the algorithm 1.*

4. *Conclude:*

   (a) *If* $\rho(\{R\}) < 1$ *and* $\nu(\{R\}) < 1$*, the model is determinate and the stable equilibrium is given by equation* (21)

   (b) *If* $\rho(\{R\}) < 1$ *and* $\nu(\{R\}) > 1$*, the model admits many stable equilibria.*

   (c) *If* $\rho(\{R\}) > 1$ *and* $\nu(\{R\}) < 1$*, the model admits no stable equilibria.*

   (d) *If* $\rho(\{R\}) > 1$ *and* $\nu(\{R\}) > 1$ *return to step 1 and try to find another solution to the matricial equation* (18) *by using another solving techniques (Foerster et al., 2016; Maih, 2015, for instance).*

The Matlab codes of algorithm 2 are available online. It is worth noticing that, in the economic models we have studied up to now, we have never encountered case (d). We use this algorithm in section 6.

4 Classes of equilibria and indeterminacy

In the absence of regime switching, restriction of the stochastic properties of the equilibria, such as their correlation with past fundamental shocks, does not affect stability. By contrast, in a regime-switching economy, the stability of equilibria depends on their co-movement with past regimes. By ruling out some classes of equilibria, the existing literature thus underestimates the size of the indeterminacy region.

In this section, we construct some sunspot equilibria that have never been described before and that appear when Proposition 2 is not verified. This illustrates the sensitivity of determinacy conditions to restrictions related to the class of equilibria and argues in favor of an intrinsic determinacy condition as in Proposition 5 that does not arbitrarily restrict the solution space. For the sake of simplicity, we only consider purely forward-looking models (13), the extension to predetermined variables in subsection 3.3 can be used to construct sunspots for more general models.

4.1 Sunspot equilibria

Here we generalize the approach by Farmer et al. (2010b). We focus on the class of equilibria, $\mathcal{M}_q$, that depend on lagged variables and regimes of this form:[11]

$$z_t = z_t^F + V_t(s_{t-q} \cdots s_t) w_t,$$

$$w_t = \phi_t(s_{t-q}, \cdots, s_t) w_{t-q} + \xi_t,$$

where $V_t(.)$ is a time-varying matrices depending on q last regimes and that belongs to a single vector every q regimes. The scalar $\phi_t(.)$ is time-varying and the product of $\phi(.)$ along a trajectory of q successive regimes is smaller than one. The sunspot shock ξ_t is a zero mean i.i.d. shock. Finally, we assume that w_t is zero for $t \leq 0$. This type of solution corresponds to recursive solutions which cyclically belong to the same one-dimensional space and are exponentially decreasing on it.

We provide determinacy conditions for this particular subspace, $\mathcal{M}_q$, in Proposition 6. We call a non fundamental solution in $\mathcal{M}_q$ a sunspot equilibrium of order q.

Proposition 6. *If for a certain integer q, there exist N^{q+1} real numbers in the open unit disk, $\alpha(i_0, \cdots, i_q)$, such that the highest eigenvalue of the matrix*

$$\left[\sum_{(i_1, \cdots, i_{q-1}) \in \{1,N\}^{q-1}} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{q-1} j} \Gamma_i^{-1} \cdots \Gamma_{i_{q-1}}^{-1} \alpha(i, i_1, \cdots, i_{q-1}, j) \right]_{(i,j)} \tag{26}$$

[11] Class of equilibria considered by Davig and Leeper (2007), Farmer et al. (2010b) or Cho (2015) can be put into these forms.

is larger than 1*, then there exist multiple bounded solutions in* $\mathcal{M}_q$*.*

This Proposition leads then to an algorithm easy to implement. First, we fix an integer q (not too large). Then equation (26) defines a function of 2^{q+1} real numbers in the open unit disk, for which we compute the maximum. When q increases, the cost of computation is exponentially increasing.

We prove this Proposition by constructing a continuum of stable equilibria in Appendix G. For a given length, q, the equilibria we consider depend on the q past regimes, shocks and endogenous variables. These equilibria belong to a fixed span every q periods, and are exponentially decreasing. We notice that, when q increases, the size of the state variables needed to describe the equilibrium increases.

As part of our reasoning for this Proposition, let us first consider the special case of a specific q-regime trajectory, $(k_0, \cdots, k_{q-1}, k_0)$, which is sufficient to generate a larger-than-one eigenvalue of the matrix defined by equation (26).[12] In this case, there exists a vector u such that $p_{k_0k_1}p_{k_1k_2}\cdots p_{k_{q-1}k_0}\Gamma_{k_0}^{-1}\cdots\Gamma_{k_{q-1}}^{-1}u = \lambda u$, with $\lambda > 1$. Suppose that the economy is initially in regime k_0 and that expectations z_t^e belongs to the unstable eigenvector, u. In addition, suppose that all the future expectations are zero except along the regimes' trajectory $(k_0, \cdots, k_{q-1}, k_0, \cdots, k_{q-1}, k_0)$. Then, according to equation (16), the expectations decrease exponentially every q periods. Hence, economic agents can form non-zero expectations consistent with converging expectations. Therefore, the economy is indeterminate. More generally, Proposition 6 proves that indeterminacy may arise in a more general context where such a trajectory does not exist. It may happen when a combination of trajectories leads to a larger-than-one eigenvalue. In this case, non-zero expectations are consistent with converging expectations in many different regimes trajectories.

A generalisation of this proposition was recently advanced by Ogura and Jungers (2014). Basically, these authors refine our results by replacing the weights, $\alpha(i, i_1, ..., i_{q-1}, j)$, in formula (26), with particular unitary matrices.[13]

The classes of equilibria that we consider in Proposition 6 encompass those considered in the literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2015; Foerster et al., 2016). The equilibria put forward after Farmer et al. (2009b) correspond to the specific case of $q = 1$. When $q = 0$ we find back the regime dependent solution space of Davig and Leeper

[12] This corresponds to $\alpha(i_0, \cdots, i_q) = \delta_{k_0i_0}\cdots\delta_{k_0i_q}$ in equation (26), where $\delta_{ij} = 1$ when $i = j$.

[13] In their construction, the vector of endogenous variables belongs to a multi-dimensional space every q periods.

(2007). By focusing on smaller classes of equilibria, the literature underestimates the size of the indeterminacy region. In turn, determinacy condition put forward in section 3 takes into account all kinds of sunspot equilibria.[14]

Proposition 6 also allows us to link our approach to the one with an initial condition and a starting date, helping interpreting indeterminacy in the context of regime switching. When Proposition 6 proves that the model is indeterminate, Lemma 4 gives examples of distinct stable equilibria solution of the model for any date greater than an initial date t_0 that we can reinterpret as an initial condition ($t_0 = 0$). According to this Lemma, the model admits at least two different equilibria:

- A fundamental solution $z_t^1 = z_t^F$
- A non-fundamental solution $z_t^2 = z_t^F + w_t$, where w_t is given in Lemma 4 Appendix G: it depends on w_{t-1} and a sunspot component ξ_t independent of the history of regimes s^t.

By assuming that $w_t = 0$ for all $t \leq 0$ and $w_0 = V(s_0)\xi_0$ with ξ_0 being any real-valued scalar, we see that the two solutions only differ from date-0 and onwards because of the sunspot component ξ_0. The proposition is thus able to show that whenever there is indeterminacy, there exists multiple initial conditions ($w_0 = 0$ and $w_0 \neq 0$) at least in one regime. Interestingly, indeterminacy does not necessarily mean that there are multiple initial conditions for any initial regime, in contrast with the no regime switching case, in which when the model is indeterminate there are always multiple initial conditions.[15]

4.2 Illustration in the monetary regime switching

We apply Proposition 6 to analyse the relationship between determinacy conditions and the restriction of the class of equilibria. Figure 3 displays different determinacy frontiers

[14] It is worth noting that some history dependent equilibria do not belong to $\mathcal{M}_q$ with a finite q.

[15] To get a simple intuition of this difference, consider a two-regime model with absorbing states. Suppose that the first regime corresponds to a determinate model (when taken in isolation) while the second corresponds to an indeterminate model (when taken in isolation). According to our definitions, we will say that the regime switching model is indeterminate. But if the economy is initially in the first regime, there exists a unique initial condition (and a unique path of the economy), while multiple initial conditions if the economy is initially in the second regime.

depending on the solution space: the whole solution space (the plain line), the regime dependent equilibria, $\mathcal{M}_0$ (the line with crosses), and the sunspot equilibria of order q built in Proposition 6 (the dashed lines).

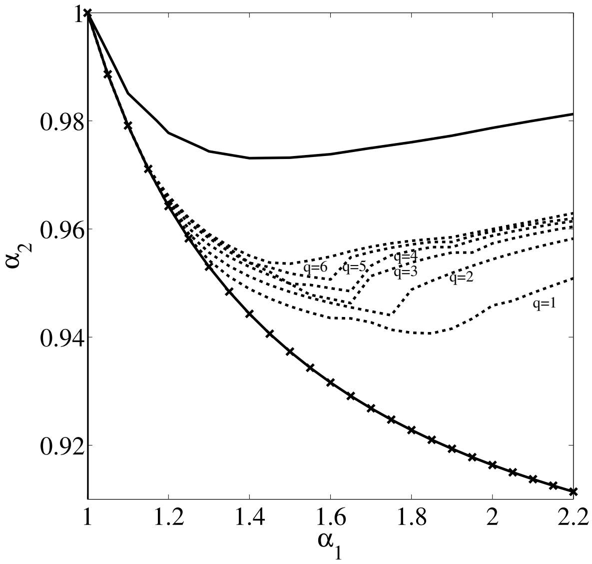

Figure 3: Existence of sunspots of order q

Note: The thick line displays the determinacy frontier (constructed as in figure 1), dashed lines correspond to sufficient indeterminacy conditions given by Proposition 6 for sunspot equilibria of order q*. The line with crosses represents the Davig and Leeper (2007) determinacy condition. Probabilities are set to* $p_{11} = 0.8$ *and* $p_{22} = 0.95$.

The sufficient indeterminacy frontier nears the determinacy condition computed as in Section 3.2 when the order of the sunspot equilibria q increases. The diminution of the determinacy region when going from $q = 0$ to $q = 1$, that is to say when adopting the solution space of Farmer et al. (2010b), is substantial. However, pushing to a higher order is not negligible either, implicitly proving the existence of multiple solutions of order larger than 1 even if there exists a unique bounded equilibrium in $\mathcal{M}_1$.

The existing literature increasingly adopts the restriction first proposed by Farmer et al. (2009b) consisting of $\mathcal{M}_1$ and sometimes referred to as Minimum State Variables solutions. Restricting the class of equilibria results in underestimating the size of the indeterminacy region. In section 5, we provide economic results that are valid for the broadest solution space.

5 The Taylor principle with regime switching

In this section we discuss the usefulness of the Taylor principle as a guideline for monetary policy in a context of regime switching. We put forward two main results. First, we revisit one of the main findings of the paper by Davig and Leeper (2007). They show that determinacy is not necessarily generated by a deviation from the Taylor principle if this deviation is small and short lasting. We prove that the result remains valid even if we consider all bounded equilibria. Thus, the criticism by Farmer et al. (2010a) does not invalidate this result even if the occurrence of such policy configurations is substantially smaller than what was predicted by the Long Run Taylor Principle initially put forward.

Second, we show that indeterminacy can arise even if the two monetary policy regimes satisfy the Taylor principle. This finding has also been noted by Foerster (2016) and Cho (2015) in separate and independent works. However, we find that such configurations arise in more cases than what is put forward in these papers, because we do not restrict the solution space. Finally, we give intuitions on why such policy configurations lead to indeterminacy and draw some policy conclusions.

In this section, we assume that all the non-policy parameters are unchanged across regimes. For policy parameters, we assume that the interest rate only reacts to inflation. This assumption allows for an economic interpretation of our determinacy results without loss of generality.

5.1 The Taylor principle is neither necessary...

Result 1. *A regime-switching economy may be determinate even if one of the regimes does not satisfy the Taylor principle.*

Monetary policy may deviate from the Taylor principle moderately ($\alpha_1 = 0.98$) but relatively persistently ($p_{11} = 0.95$) without implying indeterminacy if monetary policy reacts sufficiently to inflation in the other regime (for instance if $\alpha_2 = 1.5$) as shown in Figure 1. While the first regime is not active enough to ensure determinacy on its own, expectations of a switch towards a more active regime are enough to anchor expectations, and hence to rule out indeterminacy. Thus, expectations of a more aggressive monetary policy may be effective in guaranteeing macroeconomic stability.

This result definitively proves the Davig and Leeper (2007) claim that "a unique bounded equilibrium does not require the Taylor principle to hold in every period". In this paper, the authors obtain this result by restricting the solution space. The counter-example given in Farmer et al. (2010a) could suggest that this result does not hold when considering a broader solution space. We prove that Davig and Leeper (2007)'s result holds even if we consider the whole class of equilibria, and hence, is not an artifact due to the specific solution space they consider.

However, compared to the paper Davig and Leeper (2007), the occurrence of such policy configurations is less frequent than initially conjectured as displayed in Figure 1. Only small and brief deviations from the Taylor principle do not endanger determinacy issues.

5.2 ...nor sufficient

Result 2. *An economy may suffer from indeterminacy even if the two monetary policy regimes satisfy the Taylor principle.*

We identify such a configuration when the two monetary policies share the same rule - the nominal interest rate reacts proportionally to inflation in both regimes- but with different intensities. The two regimes satisfy the Taylor principle. In the first regime, the less active one, the central bank reacts moderately to inflation ($\alpha_1 = 1.01$). In the second, the more active regime, the central bank reacts (extremely) strongly to inflation ($\alpha_2 = 6$). Proposition 6 proves that the economy is indeterminate when monetary policy switches between prolonged periods of the less active regime ($p_{11} = 0.95$) and short-lasting periods of the more active regime ($p_{22} = 0.5$).

Figure 8 in Appendix plots the ten largest contributions to sequence u_k, which measures the convergence of expectations. The larger the contribution, the more converging the expectations along the regimes trajectory. Unsurprisingly, a prolonged less active monetary policy regime significantly contributes to an increase in this measure of stability. Indeed, in this regime, monetary policy causes expectations to diverge but only slowly as the response to inflation is weak in this regime. This regimes trajectory is not enough however to explain indeterminacy by itself as the less active regime satisfies the Taylor principle, and hence, induces determinacy when taken in isolation. The other greatest contributions correspond to the alternation between a short period of the more active regime and a protracted period of the less active regime. We explain in Figure 4 why these regimes trajectories explain indeterminacy.

Indeterminacy arises when expectations can be non-zero without generating diverging expectations. Figure 4 reports the responses of the output gap and inflation to a 17-quarter-

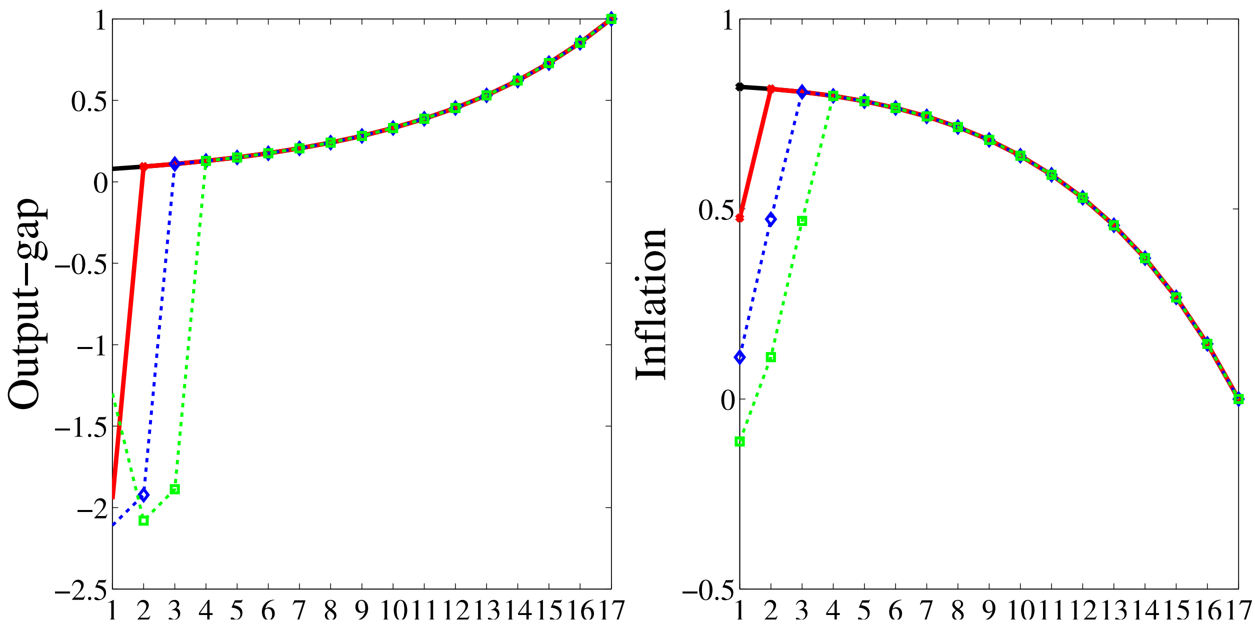

Figure 4: Impact of a positive output gap expectation in a new-Keynesian model with Markov-switching monetary policy

Note: the two figures report the dynamics of the output gap (left) and inflation (right) expectations to an expectation of an increase in the output gap in 17 periods. We relate the four largest contributors to u_{16} *(see Figure 8). The bold line represents the trajectories of inflation and the output gap conditional on staying in the less active regime (regime* 1*). Along the plain line with crosses (the dashed line with diamonds and the dashed line with squares), we plot the trajectories when the economy is in the more active regime during one (two and three resp.) period (regime* 2*) and in the less active regime afterward (regime* 1*). Policy parameters and probabilities are set to* $\alpha_1 = 1.01$, $\gamma_1 = 0$ *and* $p_{11} = 0.95$ *and* $\alpha_2 = 6$, $\gamma_2 = 0$ *and* $p_{22} = 0.5$.

ahead expectation conditional on the future regimes path. We plot the responses to an expected increase in the output gap ($\mathbb{E}_t y_{t+17} = 1$ while $\mathbb{E}_t \pi_{t+17} = 0$) in 17 quarters. If the model is determinate such expectations should necessarily lead to small expectations in the first period (when weighted by the regime trajectory's probability) as non-zero expectations always diverge in a determinate economy.

Along the line without markers, we plot the expectations dynamics if the economy remains in the less active regime. In this regime, a positive expectation of the output gap in the remote future leads to inflationary pressure with the new-Keynesian Phillips curve explaining the downward dynamics of inflation expectations. On the other hand, these inflation expectations lead to a moderate increase in the real rate that contributes to a slightly moderate output gap through the IS-curve. This means that output gap expectations increase up to the final period. If the period of less active monetary policy lasts longer, then inflation expectations are finally completely stabilised through positive real rates as the Taylor principle is satisfied in this regime. Hence, inflation expectations are positive in the first period only because the duration of the sample is too short.

Along the plain line with crosses, the dashed line with diamonds and the dashed line with squares, we plot the expectations dynamics when the more active regime lasts one, two, and three periods respectively, and is followed by a long-lasting period of the less active regime. The long-lasting period of the less active regime coincides with the case described above (the lines without markers). Hence, in the more active monetary policy, the monetary authorities face positive inflation expectations and small output gap expectations and react by raising the nominal interest to dampen contemporaneous inflation. The flip-side of this policy is a large fall in the output gap. In the end, we see that period-1 expectations of the output gap are greater than period-17 output gap expectations while inflation expectations are close to zero in both cases. This means that conditional on these regimes trajectories, non-zero expectations in the first period are consistent with converging expectations. If such regime successions occur with sufficiently large probabilities, multiple equilibria eventually arise.

This configuration is more likely to occur when the more active monetary policy regime is infrequent and short-lasting. Conversely, indeterminacy occurs when the less active monetary policy regime is very persistent. Figure 9 in Appendix displays determinacy regions with respect to the probabilities in each regime. Thus, contrary to what one might expect, expectations of infrequent, highly active monetary policy regimes may introduce converging non-zero expectations rather than increasingly diverging expectations, eventually leading to indeterminacy.

For two active monetary policy regimes to trigger indeterminacy, price stickiness is crucial. In a flexible-price economy ($\kappa \to \infty$), inflation is determined only by inflation expectations since the feedback force between output and inflation observed in Figure 4 no longer holds. If a monetary policy satisfies the Taylor principle in the two regimes, current and future monetary policies (whatever the regime) cause any non-zero inflation expectations to explode.[16] Hence, non-zero inflation expectations can be discarded and inflation is uniquely defined. This is the reason why Result 2 does not hold in a Fisherian model of inflation determination. More

[16]In this environment, determinacy conditions depend on a simple combination of transition probabilities and inflation reaction in both regimes (see Farmer et al., 2009a).

generally, the lesson is that the larger the model, the more we should expect parameters configurations allowing for indeterminacy while each regime induces determinacy when taken in isolation. Although we formally prove Result 2 using an implausible calibration, this latter may therefore well be more worrying in more realistic models.

5.3 'Neither too weak nor too strong'

One of the consequences of the two previous results is to potentially place some restrictions on the admissible monetary policy in a regime switching economy. Consider a central banker who would like to prevent the economy from indeterminacy as this can be an uncontrollable source of volatility. If the central banker knows that he will be replaced by another central banker occasionally, determinacy conditions will generate upper and lower bounds on his degree of reaction to inflation. We assume that monetary policy reacts moderately to inflation in the most frequent and long-lasting regime ($p_{11} = 0.95$ and $\alpha_1 = 0.99$) while the other regime is short-lasting ($p_{22} = 0.5$). Restrictions on admissible policies are summed up below.

Too weak or too strong a response to inflation in the infrequent regime leads to indeterminacy, while an intermediate response to inflation stabilises the economy. Figure 5 shows determinacy regions with respect to the inflation response in the infrequent regime. The grey area represents the indeterminacy region and the white area the determinacy region. Determinacy arises for responses to inflation, α_2, between at least 1.1 and 3.6 (white area).

This result stems from the combination of two effects. First, a large enough reaction to inflation helps to rule out indeterminacy arising due to the passive monetary policy in the long-lasting regime. By raising the nominal interest rate when facing inflationary pressure, the central banker helps reduce the interplay between inflation and inflation expectations (result 1). Second, an excessively strong response to inflation in one monetary policy regime increases the sensitivity of the output gap to expectations and may eventually lead to indeterminacy (Result 2).

Thus, when choosing their policies, central bankers should internalise the possibility of a switch to a passive monetary policy and thus be forced to moderate their reaction compared to what it would otherwise have been.

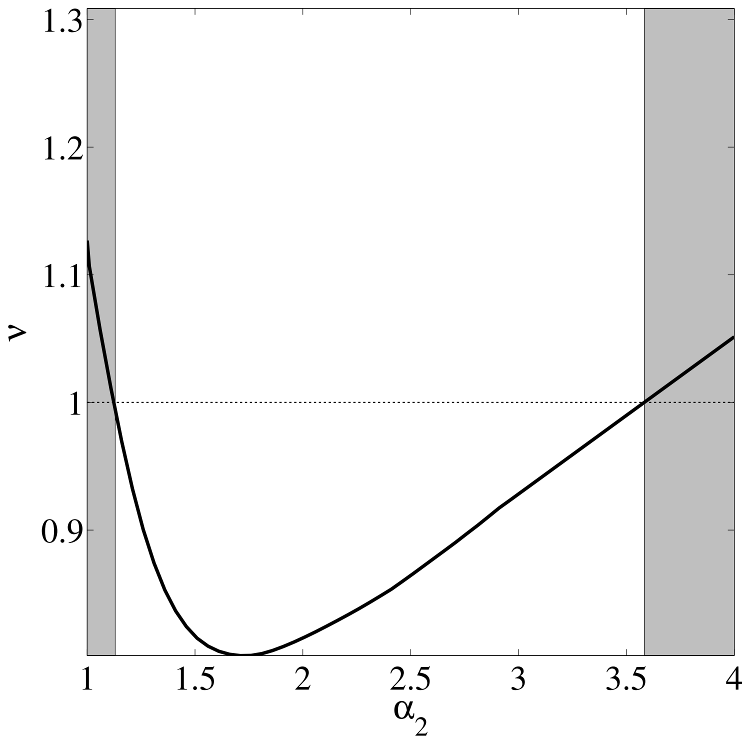

Figure 5: Which degree of activism? 'Nor too weak neither too strong'

Note: the white area represents the determinacy region with respect to the response to inflation in regime 2; the light-shaded area represents a region in which there exists multiple bounded equilibria. Policy parameters are calibrated as follows: $\alpha_1 = 0.99$, $p_{11} = 0.95$ *and* $p_{22} = 0.5$. *The dark line shows the index of stability,* u_{20} *for an optimized norm* $||.||_{Q_5^*}$. *When this line is below one the economy is determinate. Otherwise, it is indeterminate.*

6 Indeterminacy and the US Great Inflation.

In this section, we show that indeterminacy can explain the US Great Inflation period in the 70s as long as economic agents did not anticipate a regime switch with a sufficiently high probability.

We base our analysis on estimates by Lubik and Schorfheide (2004). According to these authors, the failure of the Taylor principle in the 70s explains the Great Inflation in the 70s (see also Clarida et al., 2000), while the post-Volcker active monetary policy explains the Great moderation afterward. We show that this result is very sensitive to private agents' expectations about future policies. Indeterminacy appears only if private agents expect to remain in the passive regime with a probability higher than 0.93-0.94. The transition probability from the active regime to the passive regime does not play an important role for determinacy conditions. This probability remains crucial to ensure the isolation of the post-Volcker regime to potential sunspot shocks in the other regime.

Calibration We calibrate all parameters following Lubik and Schorfheide (2004).[17] Their estimates are obtained in the specific case where regimes are permanent, $p_{11} = p_{22} = 1$. Compared to the previous section, their estimates exhibit changes in structural parameters and persistence in the Taylor rule but the model can be written as in Equation (10). We do not take into account correlation between shocks and the autocorrelation of shocks as they do not affect indeterminacy. According to these estimates, the response to inflation in the first regime is 0.89 while it is 2.19 in the second. The Taylor principle is thus only satisfied in the second regime, i.e. the Post-Volcker sample.

Counterfactual exercise We investigate to what extent the indeterminacy-based explanation of the 70s US Great Inflation depends on agents' beliefs about transition probability toward a more active monetary policy regime. We posit that private agents have never expected regime switching in the past (as in Lubik and Schorfheide, 2004). Therefore we assume that the estimated parameters would have remained unchanged assuming different transition probabilities. We then conduct a counterfactual exercise and compute determinacy conditions under different scenarios on transition probabilities. This counterfactual exercise resembles Bianchi (2013)'s paper. In this paper, the author studies dynamic properties of the US postwar economy by assuming determinacy and running counterfactual simulations on beliefs about future policy regimes.

Results The determinacy frontier is highly sensitive to the persistence of the pre-Volcker regime. Figure 6 plots the determinacy region (in white) and the indeterminacy region (in dark grey) depending on the probabilities to remain in the pre-Volcker regime (p_{11}) and in the post Volcker regime (p_{22}). Since the Taylor principle is not satisfied in the first regime, the economy is obviously indeterminate for $p_{11} = p_{22} = 1$. Indeterminacy disappears as soon as the persistence of the pre-Volcker regime is sufficiently small, $p_{11} < 0.94$. The probability to remain in the post-Volcker regime (p_{22}) is not crucial for determinacy. These features imply that the explanation of the US Great Inflation based on indeterminacy relies on the

[17] We calibrate all the parameters following Table 3 (Prior 2) of Lubik and Schorfheide (2004). Using Prior 1 instead of Prior 2 does not change qualitatively the results.

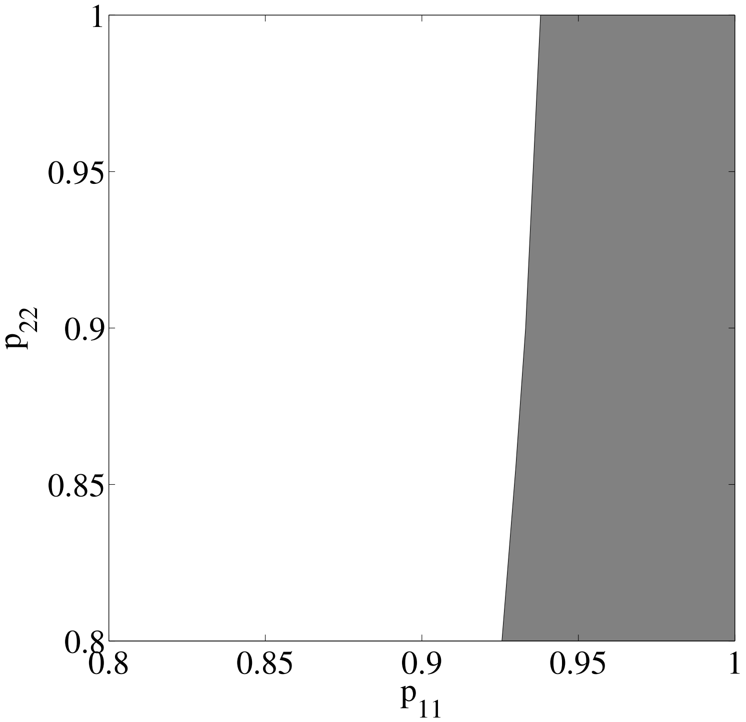

Figure 6: What if economic agents had expected regime switching?

Note: the white area represents the determinacy region with respect to the probability to stay in the pre-Volcker regime, p_{11} *and in the post-Volcker regime,* p_{22}*. The rest of the parameters are calibrated following Lubik and Schorfheide (2004). The darkest region shows the indeterminacy region computed using Proposition 3. We compute the index of stability,* u_{20} *for an optimized norm* $||.||_{Q_5^*}$ *following algorithms 1 and 2.*

assumption that the probability to remain in this regime was sufficiently high.

Furthermore, the estimated probabilities in the literature[18] include transition probabilities consistent with indeterminacy. This suggests that, given current empirical studies, we cannot refute the indeterminacy-based explanation by arguing that economic agents expected regime switching. Besides, if economic agents believe that the post-Volcker regime is not absorbing, it means that this latter regime is not isolated from the passive monetary policy regime and can suffer from extrinsic volatility.

Finally, a further investigation into the private agents' exact beliefs during the 70s would be crucial to settle whether indeterminacy can explain the Great Inflation or not. To do

[18] For instance, Bianchi (2013) estimate the 90% confidence interval for this probability between 0.83 and 0.96 with a posterior mode at 0.93. While they estimate a slightly different model, recent estimates by Baele et al. (2015) also suggest a relatively high persistence of the passive monetary regime.

so, one needs to estimate the model and transition probabilities simultaneously allowing for determinacy and indeterminacy in the spirit of Baele et al. (2015). This may be a rich avenue for future research but a challenging one since the structure of sunspot shocks can be extremely complex.

7 Conclusion

From a theoretical standpoint, this paper establishes a necessary and sufficient condition of determinacy for rational expectations models with Markov-switching. This condition depends on the asymptotic behavior of all matrix products. The complexity raised by regime-switching models reflects the path-dependency of economic agents' expectations. Hence, determinacy conditions in the presence of regime switching depend on the restrictions of the solution space. To overcome this difficulty, we derive an efficient algorithm to check determinacy in practice.

We then generalise the Taylor principle to a canonical monetary model in which monetary policy switches between different Taylor rules. We establish a non-trivial relationship between monetary policy regimes and determinacy. On the one hand, an active monetary policy may help anchor inflation expectations if the other regime fails to satisfy the Taylor principle. On the other hand, an over-active monetary policy may lead to indeterminacy. This aggressive monetary policy destabilises the output gap by over-reacting to inflation expectations due to the other regime. This may happen even if the two regimes satisfy the Taylor principle. This last result emphasises the fact that, even if the different regimes ensure determinacy when taken in isolation, regime switching itself may destabilise the economy, should these regimes be very different. We thus identify situations in which policy switching itself, rather than one or the other policy, is responsible for indeterminacy.

Finally, we revisit the explanation of Great Inflation based on indeterminacy. We find that this argument is sensitive to agents' beliefs about future regimes and especially the probability related to a regime switch. The determinacy frontier with respect to transition probabilities is very close to estimated probabilities in the existing literature. A careful examination of these beliefs is thus required to confirm or deny this explanation.

Additional Figures

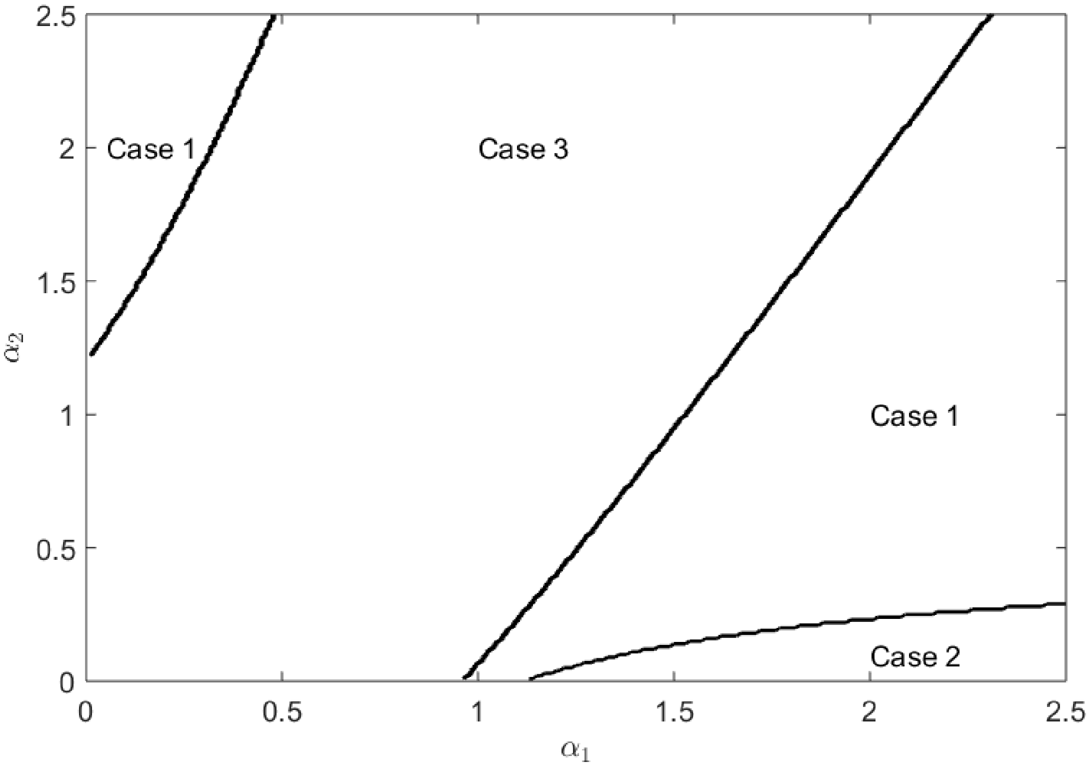

Figure 7: Illustration of Proposition 3: determinacy and indeterminacy regions when Taylor rules respond to past inflation

Note: We represent the different cases defined in Proposition 3 with respect to the responses of the nominal interest rate to current inflation in regime 1 α_1 *and in regime 2* α_2*. We calibrate other parameters as follows:* $\mu_1 = 0.2$, $p_{11} = 0.5$, $\mu_2 = 1.2$, $p_{22} = 0$*. Case 1 corresponds to the region of parameters that is consistent with a unique equilibrium (determinacy), Case 2 with multiple equilibria (indeterminacy), and Case 3 with no equilibria.*

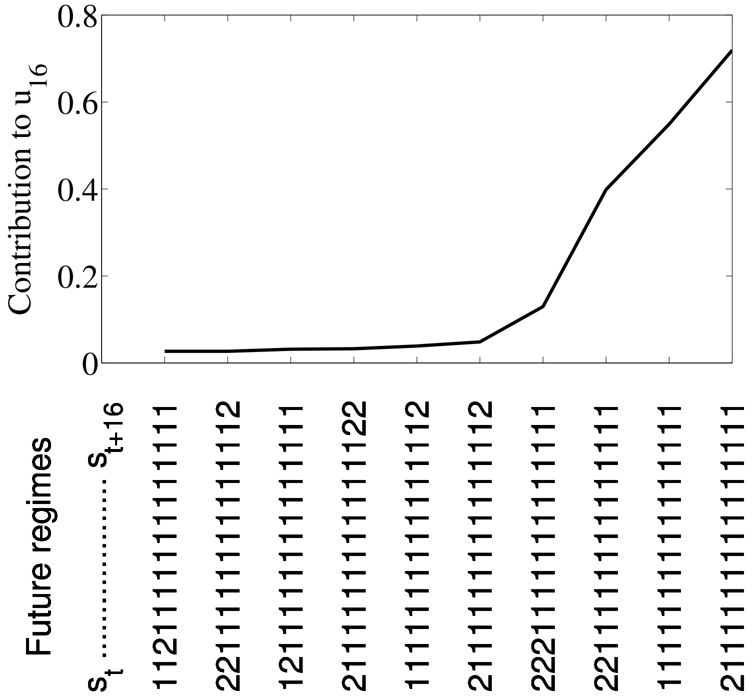

Figure 8: Ten largest contributions to a measure of expectations explosiveness

Note: we report the ten largest contributions -in terms of the future regime trajectory and amongst 2^{36} *possible trajectories- to a measure of model stability,* u_{16} *(see Proposition 1). The x-axis represents a particular regimes trajectory. The y-axis stands for the associated contribution to* u_{16}*. A higher contribution suggests that expectations diverge less along this regimes trajectory. When all the contributions add up to less than one, this proves that the economy is determinate. Otherwise, it suggests (without formally proving) indeterminacy. Finally, if one contribution is larger than one, the economy is indeterminate and we can build converging non-zero expectations along this regimes trajectory. Policy parameters and probabilities are set to* $\alpha_1 = 1.01$*,* $\gamma_1 = 0$ *and* $p_{11} = 0.95$ *and* $\alpha_2 = 6$*,* $\gamma_2 = 0$ *and* $p_{22} = 0.5$*.*

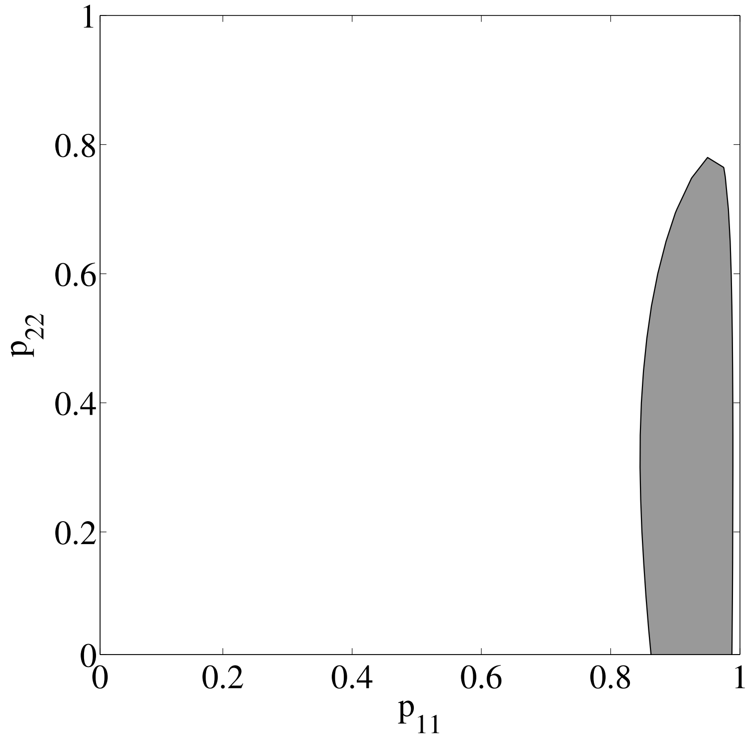

Figure 9: Determinacy regions and persistence of regimes: new-Keynesian model with Markov-switching monetary policy

Note: the white area represents the determinacy region with respect to probability of remaining in each regime; the light-shaded area a region in which there exist multiple bounded equilibria. Policy parameters are calibrated such that the two regimes satisfy the Taylor principle. Response to inflation in the first (second) regime is $\alpha_1 = 1.01$ *(*$\alpha_2 = 6$*, respectively).*

References

Baele, L., G. Bekaert, S. Cho, K. Inghelbrecht, and A. Moreno (2015): "Macroeconomic regimes," *Journal of Monetary Economics*, 70, 51–71.

Barthélemy, J. and M. Marx (2017): "Solving endogenous regime switching models," *Journal of Economic Dynamics and Control*, 77, 1–25.

Bianchi, F. (2013): "Regime Switches, Agents' Beliefs, and Post-World War II U.S. Macroeconomic Dynamics," *Review of Economic Studies*, 80, 463–490.

Blanchard, O. and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48.

Branch, W., T. Davig, and B. McGough (2007): "Expectational Stability in Regime-Switching Rational Expectations Models," *Federal Reserve Bank of Kansas City Research Working Paper*.

Cho, S. (2015): "Sufficient conditions for determinacy in a class of Markov-switching rational expectations models," *Review of Economic Dynamics*, –.

Clarida, R., J. Galí, and M. Gertler (2000): "Monetary Policy Rules And Macroeconomic Stability: Evidence And Some Theory," *The Quarterly Journal of Economics*, 115, 147–180.

Conway, J. (1990): *A Course in Functional Analysis*, Springer.

Costa, O., M. Fragoso, and R. Marques (2005): *Discrete-Time Markov Jump Linear Systems*, Springer.

Davig, T. and E. M. Leeper (2007): "Generalizing the Taylor Principle," *American Economic Review*, 97, 607–635.

——— (2010): "Generalizing the Taylor Principle: Reply," *American Economic Review*, 100, 618–624.

Farmer, R. E. A., D. F. Waggoner, and T. Zha (2009a): "Indeterminacy in a Forward-Looking Regime Switching Model," *International Journal of Economic Theory*, 5, 69–84.

——— (2009b): "Understanding Markov-switching rational expectations models," *Journal of Economic Theory*, 144, 1849–1867.

——— (2010a): "Generalizing the Taylor Principle: a Comment," *American Economic Review.*

——— (2010b): "Minimal State Variable Solutions to Markov-Switching Rational Expectations Models," *in Journal of Economic Dynamics and Control.*

Foerster, A., J. Rubio-Ramirez, D. Waggoner, and T. Zha (2016): "Perturbation Methods for Markov-switching DSGE models," *Quantitative economics.*

Foerster, A. T. (2016): "Monetary Policy Regime Switches And Macroeconomic Dynamics," *International Economic Review*, 57, 211–230.

Jin, H. and K. Judd (2002): "Perturbation Methods for General Dynamic Stochastic Models," *Working Paper, Stanford University.*

Jungers, R. and V. Protasov (2011): "Fast Algorithm for the p-radius Computation," *SIAM Journal on Scientific Computing*, 33(3), 1246–1266.

Lubik, T. A. and F. Schorfheide (2004): "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review*, 94, 190–217.

Maih, J. (2015): "Efficient perturbation methods for solving regime-switching DSGE models," *Working Paper, Norges Bank.*

Ogura, M. and R. Jungers (2014): "Efficiently Computable Lower Bounds for the p-radius of Switching Linear Systems," *53rd IEEE Conference on Decision and Control (Los Angeles, USA, 2014), Proceedings of CDC 2014.*

Sims, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20, 1–20.

Taylor, J. B. (1993): "Discretion versus Policy Rules in Practice," *Carnegie-Rochester Conference Series on Public Policy*, 39.

Theys, J. (2005): "Joint Spectral Radius: Theory and Approximations," *PhD Thesis, Center for Systems Engineering and Applied Mechanics, Université Catholique de Louvain.*

UHLIG, H. (1999): "Analysing non Linear Dynamic Stochastic Models," *Computational Methods for the Study of Dynamic Economies, in R. Marimon and A. Scott (eds), Oxford University Press.*

VANKEERBERGHEN, G., J. HENDRICKX, AND R. M. JUNGERS (2014): "JSR: A Toolbox to Compute the Joint Spectral Radius," in *Proceedings of the 17th International Conference on Hybrid Systems: Computation and Control*, New York, NY, USA: ACM, HSCC '14, 151–156.

WOODFORD, M. (1986): "Stationary Sunspot Equilibria: The Case of Small Fluctuations around a Deterministic Steady State," *mimeo.*

——— (2003): "Interest and prices: Foundations of a Theory of Monetary Policy," *Princeton University Press.*