Supplement to “Identification of time and risk preferences in buy price auctions”: Appendix
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This supplementary appendix contains proofs for some results in the paper and additional discussion of identification of utility functions from certainty equivalents.

Appendix SA: Proofs of selected results from the main text

Proof of Proposition 2

Consider the term
\[ e^{-\gamma U(v-r)} + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma n}}{n!} F^n_V(v) E_n[U(v - \max\{r, Y\})|Y \leq v]. \]  
(SA.1)

Note that
\[
F^n_V(v) E_n[U(v - \max\{r, Y\})|Y \leq v] = \int_0^v U(v - \max\{r, y\}) n F^{n-1}_V(y) f_V(y) \, dy
= \int_0^r U(v - r) n F^{n-1}_V(y) f_V(y) \, dy + \int_r^v U(v - y) n F^{n-1}_V(y) f_V(y) \, dy
= U(v - r) F^n_V(r) + \int_r^v U(v - y) n F^{n-1}_V(y) f_V(y) \, dy.
\]
So we can write (SA.1) as

\[
e^{-\gamma} U(v - r) + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} e^{-\gamma} U(v - r) F_V^n(r) + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int_r^v U(v - y) n F_V^{n-1}(y) f_V(y) \, dy
\]

\[
e^{-\gamma} U(v - r) \left[ 1 + \sum_{n=1}^{\infty} \frac{\gamma^n F_V^n(r)}{n!} \right] + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int_r^v U(v - y) n F_V^{n-1}(y) f_V(y) \, dy
\]

\[
e^{-\gamma} U(v - r) \exp(\gamma F_V(r)) + e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int_r^v U(v - y) n F_V^{n-1}(y) f_V(y) \, dy
\]

\[
= U(v - r) \exp[\gamma F_V(r) - \gamma] + e^{-\gamma} \int_r^v U(v - y) f_V(y) \left[ \sum_{n=1}^{\infty} \frac{\gamma^n F_V^{n-1}(y)}{n!} \right] \, dy,
\]

where the last equality follows from the dominated convergence theorem. We also have

\[
\sum_{n=1}^{\infty} \frac{\gamma^n}{n!} F_V^{n-1}(y) = \sum_{n=1}^{\infty} \frac{\gamma^n}{(n-1)!} F_V^{n-1}(y) = \gamma \exp(\gamma F_V(y)),
\]

so

\[
U^R(v, t) = \delta(T - t) \left\{ U(v - r) \exp[\gamma F(r) - \gamma] + \int_r^v U(v - y) \exp(\gamma F_V(y) - \gamma) f_V(y) \, dy \right\}.
\]

The other parts of the proposition are straightforward to verify.

**Proof of Proposition 4**

We show properties of the inverse cutoff function defined by

\[
p(c, r, \tau, t) = c - U^{-1}\left( \delta(\tau) \left( \alpha(r, \tau, t) U(c - r) + \int_r^c U(c - y) h(y, \tau, t) \, dy \right) \right)
\]

\[
= c - M(c, r, \tau, t)
\]

over the support \( r \in [0, \infty), c \in [r, \infty), \tau \in (0, \infty), \) and \( t \in (0, \infty). \)

We start by deriving some useful properties of \( U^{-1}(x) \) and \( U^{-1\prime}(x) \) given Assumption 1. Starting with the identity

\[
z = U^{-1}(U(z)),
\]

differentiate w.r.t. \( z \) to get

\[
1 = U^{-1\prime}(U(z)) U'(z).
\]

Evaluating this expression at \( z = U^{-1}(x) \), we obtain

\[
U^{-1\prime}(x) = \frac{1}{U'(U^{-1}(x))} = (U'(U^{-1}(x)))^{-1}.
\]
Differentiating this results in

\[ U^{-1''}(x) = -(U'(U^{-1}(x)))^{-2} U''(U^{-1}(x)) U^{-1'}(x) \]
\[ = -(U'(U^{-1}(x)))^{-2} U''(U^{-1}(x))(U'(U^{-1}(x)))^{-1} \]
\[ = -(U'(U^{-1}(x)))^{-3} U''(U^{-1}(x)). \]

Given our assumptions on \( U(x) \), these results imply the following statements:

(i) We have \( U^{-1'}(0) = 1 \).

(ii) We have \( U^{-1''}(0) = -U''(0) \).

(iii) We have that \( U^{-1'}(\cdot) > 0 \) and is bounded away from 0 and \( \infty \).

(iv) We have that \( U^{-1''}(\cdot) \geq 0 \) and is bounded away from \( \infty \).

With these results in hand, consider the statements in the proposition one by one. First, \( p_c(c, r, \tau, t) > 0 \) because, by Assumption 3, the derivative of \( M \) w.r.t. its first argument is strictly less than 1.

Second, \( p_c(c, r, \tau, t) < 1 \), since

\[
p_c(c, r, \tau, t) = 1 - U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) \, dy \right) \right) \cdot \delta(\tau) \left( \alpha(r, \tau, t) U'(c-r) + \int_r^c U'(c-y) h(y, \tau, t) \, dy \right),
\]

and because under our assumptions, \( U^{-1'}(\cdot) > 0 \), \( \delta(\cdot) > 0 \), \( U'(\cdot) > 0 \), \( \alpha(y, \tau, t) > 0 \), and \( h(y, \tau, t) > 0 \) for \( y > r \).

Third, \( p_r(c, r, \tau, t) > 0 \), since

\[
p_r(c, r, \tau, t) = -U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) \, dy \right) \right) \cdot \delta(\tau) \left( \alpha(r, \tau, t) U'(c-r) - \alpha(r, \tau, t) U'(c-r) + U(c-r) h(r, \tau, t) \right) \]
\[ = U^{-1'} \left( \delta(\tau) \left( \alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) \, dy \right) \right) \]
\[ \cdot \delta(\tau) \alpha(r, \tau, t) U'(c-r). \]

The second line follows since \( \frac{\partial \alpha(r, \tau, t)}{\partial r} = h(r, \tau, t) \), and the term is strictly positive since under our assumptions, \( U^{-1'}(\cdot) > 0 \), \( \delta(\cdot) > 0 \), \( U'(\cdot) > 0 \), and \( \alpha(y, \tau, t) > 0 \).
Fourth, 
\[ p_{\tau}(c, r, \tau, t) \geq 0, \text{ since} \]
\[ p_{\tau}(c, r, \tau, t) = -U^{-1}\left( \delta(\tau)\left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t) \, dy \right) \right) \]
\[ \cdot \left[ \delta'(\tau)\left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t) \, dy \right) \right] \cdot \frac{\partial}{\partial \tau} \left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t) \, dy \right). \]

The first term in the square brackets is weakly negative since Assumption 1 implies 
\( \delta'(\cdot) < 0, \alpha(\cdot, \cdot, \cdot) > 0, h(\cdot, \cdot, \cdot) > 0, \) and 
\( U(\cdot) \geq 0. \) The second term in the square brackets is weakly negative since 
\( \delta(\tau) > 0 \) and the derivative of the expected utility from rejecting 
the BP w.r.t. \( \tau \) is weakly negative (since the distribution of the highest competitor valuation is stochastically increasing in the length of the bidding phase \( \tau; \) this derivative is zero when \( c = r. \)) Since \( U^{-1}(x) > 0, \) this implies 
\( p_{\tau}(c, r, \tau, t) \geq 0. \)

Fifth, 
\( p(c, r, \tau, t) = r \text{ iff } c = r, \) because
\[ p(c, c, \tau, t) = c - U^{-1}\left( \delta(\tau)\left( \alpha(c, \tau, t)U(c - c) + \int_c^c U(c - y)h(y, \tau, t) \, dy \right) \right) \]
\[ = c - U^{-1}(0) \]
\[ = c = r. \]

The “only if” follows because \( p_c(c, r, \tau, t) > 0 \) and because \( p(c, r, \tau, t) \) is only defined for 
\( c \geq r. \)

Sixth, 
\( p(c, r, \tau, t) \geq r \) from a similar argument, since 
\( p(c, r, \tau, t) = r \) when \( c = r \) and 
\( p_c(c, r, \tau, t) > 0. \)

Seventh, 
\( p(c, r, \tau, t) \leq c, \) since
\[ p(c, r, \tau, t) = c - U^{-1}\left( \delta(\tau)\left( \alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t) \, dy \right) \right) \]
and \( U^{-1}(\cdot) \geq 0. \)

Next, 
\( p_c(z, z, \tau, t) = 1 - \delta(\tau)\alpha(z, \tau, t), \) since
\[ p_c(z, z, \tau, t) \]
\[ = 1 - U^{-1}\left( \delta(\tau)\left( \alpha(z, \tau, t)U(z - z) + \int_z^z U(z - y)h(y, \tau, t) \, dy \right) \right) \]
\[ \cdot \delta(\tau)\left( \alpha(z, \tau, t)U'(z - z) + \int_z^z U'(z - y)h(y, \tau, t) \, dy \right) \]
\[ = 1 - U^{-1}(0)\delta(\tau)\alpha(z, \tau, t)U'(0) \]
\[ = 1 - \delta(\tau)\alpha(z, \tau, t). \]
Finally, \( p_r(z, z, \tau, t) = \delta(\tau)\alpha(z, \tau, t) \), since

\[
p_r(z, z, \tau, t) = -U^{-1'}\left( \delta(\tau)\left( \alpha(z, \tau, t) U(z - z) + \int_z^z U(z - y) h(y, \tau, t) \, dy \right) \right)
\cdot \delta(\tau)\left( \frac{\partial \alpha(z, \tau, t)}{\partial z} U(z - z) - \alpha(z, \tau, t) U'(z - z) + U(z - z) h(z, \tau, t) \right)
= U^{-1'}\left( \delta(\tau)\left( \alpha(z, \tau, t) U(z - z) + \int_z^z U(z - y) h(y, \tau, t) \, dy \right) \right)
\cdot \delta(\tau)\alpha(z, \tau, t) U'(z - z)
= U^{-1'}(0)\delta(\tau)\alpha(z, \tau, t) U'(0)
= \delta(\tau)\alpha(z, \tau, t),
\]

where the last line follows because \( U^{-1'}(0) = U'(0) = 1 \).

Next, we consider the second derivatives of the inverse cutoff function w.r.t. \( c \) and \( r \), that is, \( p_{cc}(c, r) \), \( p_{rr}(c, r) \), and \( p_{cr}(c, r) \). We drop the \( \tau \) and \( t \) arguments for compactness.

For \( p_{cc}(c, r) \), we have

\[
p_{cc}(c, r) = -U^{-1''}\left( \delta\left( \alpha(r) U(c - r) + \int_r^c U(c - y) h(y) \, dy \right) \right)
\cdot \delta\left[ \alpha(r) U'(c - r) + \int_r^c U'(c - y) h(y) \, dy \right]
= -U^{-1''}\left( \delta\left( \alpha(r) U(c - r) + \int_r^c U(c - y) h(y) \, dy \right) \right)
\cdot \delta\left[ \alpha(r) U''(c - r) + \int_r^c U''(c - y) h(y) \, dy + h(c) \right].
\]

Under our assumptions, all these terms are bounded away from \( \infty \) and \( -\infty \), so \( p_{cc}(c, r) \) is bounded away from \( \infty \) and \( -\infty \). Moreover, if we evaluate this expression at \( c = r = z \), we get

\[
p_{cc}(z, z) = -U^{-1''}\left( \delta\left( \alpha(z) U(z - z) + \int_z^z U(z - y) h(y) \, dy \right) \right)
\cdot \delta^2\left[ \alpha(z) U'(z - z) + \int_z^z U'(z - y) h(y) \, dy \right]^2
= -U^{-1''}\left( \delta\left( \alpha(z) U(z - z) + \int_z^z U(z - y) h(y) \, dy \right) \right)
\cdot \delta\left[ \alpha(z) U''(z - z) + \int_z^z U''(z - y) h(y) \, dy + h(z) \right].
\]
\[
\cdot \delta \left[ \alpha(z)U''(z - z) + \int_z^z U''(z - y)h(y) \, dy + h(z) \right] \\
= -U^{-1''}(0)\delta^2 \alpha(z)^2 - U^{-1'}(0)\delta \left[ \alpha(z)U''(0) + h(z) \right] \\
= -U''(0)\delta \alpha(z)(1 - \delta \alpha(z)) - \delta h(z).
\]

For \( p_{rr}(c, r) \), we have
\[
p_r(c, r) = U^{-1'} \left( \delta \left( \alpha(r)U(c - r) + \int_r^c U(c - y)h(y) \, dy \right) \right) \delta \alpha(r)U'(c - r),
\]
so
\[
p_{rr}(c, r) = -U^{-1''} \left( \delta \left( \alpha(r)U(c - r) + \int_r^c U(c - y)h(y) \, dy \right) \right) \delta^2 \alpha(r)^2 U'(c - r)^2 \\
+ U^{-1'} \left( \delta \left( \alpha(r)U(c - r) + \int_r^c U(c - y)h(y) \, dy \right) \right) \\
\cdot \delta \left[ \alpha'(r)U'(c - r) - \alpha(r)U''(c - r) \right].
\]

Again, under our assumptions, all the terms in this expression are bounded away from \( \infty \) and \( -\infty \), so \( p_{rr}(c, r) \) is bounded away from \( \infty \) and \( -\infty \). If we evaluate this expression at \( c = r = z \), we get
\[
p_{rr}(z, z) = -U^{-1''} \left( \delta \left( \alpha(z)U(z - z) + \int_z^z U(z - y)h(y) \, dy \right) \right) \delta^2 \alpha(z)^2 U'(z - z)^2 \\
+ U^{-1'} \left( \delta \left( \alpha(z)U(z - z) + \int_z^z U(z - y)h(y) \, dy \right) \right) \\
\cdot \delta \left[ \alpha'(z)U'(z - z) - \alpha(z)U''(z - z) \right] \\
= -U^{-1''}(0)\delta^2 \alpha(z)^2 U'(0)^2 + U^{-1'}(0)\delta \left[ \alpha'(z)U'(0) - \alpha(z)U''(0) \right] \\
= U''(0)\delta^2 \alpha(z)^2 + \delta \left[ \alpha'(z) - \alpha(z)U''(0) \right] \\
= -U''(0)\delta \alpha(z)(1 - \delta \alpha(z)) + \delta \alpha'(z).
\]

For \( p_{rc}(c, r) = p_{cr}(c, r) \), we have
\[
p_r(c, r) = U^{-1'} \left( \delta \left( \alpha(r)U(c - r) + \int_r^c U(c - y)h(y) \, dy \right) \right) \delta \alpha(r)U'(c - r),
\]
so
\[
p_{rc}(c, r) = U^{-1'} \left( \delta \left( \alpha(r)U(c - r) + \int_r^c U(c - y)h(y) \, dy \right) \right) \delta \alpha(r)U''(c - r) \\
+ U^{-1''} \left( \delta \left( \alpha(r)U(c - r) + \int_r^c U(c - y)h(y) \, dy \right) \right) \delta \alpha(r)U'(c - r) \\
\cdot \left[ \delta \left( \alpha(r)U'(c - r) + \int_r^c U'(c - y)h(y) \, dy \right) \right].
\]
Again, all the terms are bounded away from $\infty$ and $-\infty$, so $p_{rc}(c, r)$ is bounded away from $\infty$ and $-\infty$. Evaluated at $c = r = z$, we get

$$p_{rc}(z, z) = U^{-1'}\left(\delta(\alpha(z)U(z - z) + \int_z^z U(z - y)h(y)\,dy)\right)\delta\alpha(z)U''(z - z)$$

$$+ U^{-1''}\left(\delta(\alpha(z)U(z - z) + \int_z^z U(z - y)h(y)\,dy)\right)\delta\alpha(z)U'(z - z)$$

$$\cdot \left[\delta(\alpha(z)U'(z - z) + \int_z^z U'(z - y)h(y)\,dy)\right]$$

$$= U^{-1'}(0)\delta\alpha(z)U''(0) + U^{-1''}(0)\delta\alpha(z)U'(0)\delta\alpha(z)U'(0)$$

$$= \delta\alpha(z)U''(0) - U''(0)\delta\alpha(z)\delta\alpha(z)$$

$$= U''(0)\delta\alpha(z)(1 - \delta\alpha(z)).$$

**Proof of Proposition 8**

Since the hazard rate of the first action (accept or reject the BP) is observed in the data and satisfies

$$\theta(t_1| p, r, \tau_0) = \lambda(t_1)(1 - F_V(r)), \quad (SA.2)$$

it is clear that $\lambda(t_1)(1 - F_V(r))$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, \bar{T})$.

We next show that this implies that $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$. By definition

$$\alpha(r, \tau_0, t_1) = \exp(\gamma F_V(r) - \gamma),$$

where

$$\gamma = \int_t^{t+\tau_0} \lambda(s)\,ds.$$  

Therefore

$$\alpha(r, \tau_0, t_1) = \exp\left(- (1 - F_V(r)) \int_t^{t+\tau_0} \lambda(s)\,ds\right)$$

$$= \exp\left(- \int_t^{t+\tau_0} \lambda(s)(1 - F_V(r))\,ds\right).$$

Since $\lambda(t_1)(1 - F_V(r))$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$, this implies that $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

Next we show that $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$. Again, by definition

$$h(y, \tau_0, t_1) = \exp(\gamma F_V(y) - \gamma)\gamma f_V(y)$$

$$= \alpha(r, \tau_0, t_1) \int_t^{t+\tau_0} \lambda(s)f_V(y)\,ds.$$
Since $\lambda(t_1)(1 - F_V(y))$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$, its derivative $-\lambda(t_1)f_V(y)$ is also identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$. This implies that $\int_t^{t+\tau_0} \lambda(s)f_V(y)\,ds$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$. Therefore, $h(y, \tau_0, t)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

Next, we consider identification of $c(p, r, \tau_0, t_1)$. From Section 3.3, we know that

$$\Pr(B = 1|p, r, \tau_0, t_1) = \frac{1 - F_V(c(p, r, \tau_0, t_1))}{1 - F_V(r)},$$

where $\Pr(B = 1|p, r, \tau_0, t_1)$ is observed on the support $r \in [\underline{r}, \bar{r}]$, $p \in [\underline{p}, \bar{p}]$, and $t_1 \in [0, T)$ (at $\tau_0$). Therefore,

$$\Pr(B = 1|p, r, \tau_0, t_1) = \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\lambda(t_1)(1 - F_V(r))} = \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\theta(t_1|p, r, \tau_0)},$$

and therefore $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$ is identified on the same support. Note that this term is the hazard rate of the BP being accepted.

Since we have already identified $\lambda(t_1)(1 - F_V(r))$ on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$, this implies that

$$c(p, r, \tau_0, t_1) = z,$$

where $z$ satisfies

$$\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1))) = \lambda(t_1)(1 - F_V(z)). \tag{SA.3}$$

Intuitively, this says that the cutoff at $(p, r, \tau_0, t_1)$ is equal to the hypothetical reserve price that would imply that the hazard rate of the first action is equal $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$.

It remains to be verified that we can identify the $z$ that satisfies (SA.3). Note that the right-hand side (r.h.s.) of (SA.3) is strictly decreasing in $z$. Since $c(p, r, \tau_0, t_1) \geq \underline{r}$, the left-hand side (l.h.s.) is less than or equal to the r.h.s. at $z = \underline{r}$. Hence, we want to increase $z$ above $\underline{r}$ to satisfy (SA.3). The problem is that we only observe the r.h.s. for $z \in [\underline{r}, \bar{r}]$. However, as long as $c(p, r, \tau_0, t_1) \leq \bar{r}$, we can find a $z \in [\underline{r}, \bar{r}]$ that satisfies (SA.3). This implies that $c(p, r, \tau_0, t_1)$ is identified on the set $(p, r, t_1)$ such that $c(p, r, \tau_0, t_1) \leq \bar{r}$. This immediately implies that the inverse cutoff function $p(c, r, \tau_0, t_1)$ is identified on the set $r \in [\underline{r}, \bar{r}], t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \bar{r}]$.

Thus, we have shown that the following statements hold:

(i) The function $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

(ii) The function $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

\[1\]This set exists. To show this, consider a situation where $r = \underline{r}$ and $p = \underline{r} + \varepsilon$ for some arbitrarily small $\varepsilon$. For small enough $\varepsilon$, $c(p, r, \tau_0, t_1)$ will be below $\bar{r}$ (since $c$ is continuous and $c(\bar{r}, \bar{r}, \tau_0, t_1) = \bar{r}$). Obviously the size of this set will depend on the range $[\underline{r}, \bar{r}]$. 

(iii) The function \( p(c, r, \tau_0, t_1) \) is identified on \( r \in [r, \bar{r}] \), \( t_1 \in [0, T - \tau_0) \), and \( c \in [r, \bar{r}] \).

Recall that our integral equation

\[
U(c - p(c, r, \tau_0, t_1)) = \delta(\tau_0) \left( \alpha(r, \tau_0, t_1)U(c - r) + \int_r^c U(c - y)h(y, \tau_0, t_1)\,dy \right)
\]

(SA.4)
can be reduced to

\[
U''(c - r) = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)} U'(c - r),
\]

(SA.5)
where

\[
\Phi(c, r, \tau_0, t_1) = \alpha(r, \tau_0, t_1) \left[ \left( 1 - \frac{p_c(c, r, \tau_0, t_1)}{p_r(c, r, \tau_0, t_1)} \right) - 1 \right].
\]

Identification of \( \alpha(r, \tau_0, t_1) \), \( h(y, \tau_0, t_1) \), and \( p(c, r, \tau_0, t_1) \) implies that we can identify \( \Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1) \Phi(r, \tau_0, t_1) \) on \( r \in [r, \bar{r}] \), \( t_1 \in [0, T - \tau_0) \), and \( c \in [r, \bar{r}] \). Hence, by arguments similar to Proposition 3, equation (SA.5) identifies \( U(\cdot) \) on \( [0, \bar{r} - \bar{r}] \). By the same arguments as in Section 3.4, \( \delta(\cdot) \) is identified at \( \tau_0 \).

**Proof of Proposition 9**

Assumption 8 further restricts the support of \( p \) to \( [p_0 - \varepsilon, p_0 + \varepsilon] \). We also assume that \( p_0 \) is such that there exists a \( r^* \in (r, \bar{r}) \) and a \( t_1^* \) such that \( c(p_0, r^*, \tau_0, t_1^*) \in (r, \bar{r}) \). By the same arguments as in the proof of Proposition 8, we know that the following statements hold:

(i) The function \( \alpha(r, \tau, t_1) \) is identified on \( r \in [r, \bar{r}] \) and \( t_1 \in [0, \bar{T} - \tau_0) \).

(ii) The function \( h(y, \tau_0, t_1) \) is identified on \( y \in [r, \bar{r}] \) and \( t_1 \in [0, \bar{T} - \tau_0) \).

By the same arguments as above (and the condition that \( c \in (r, \bar{r}) \)), one can see that \( c(p, r, \tau_0, t_1) \) will be identified for \( p \in (p_0 - \varepsilon, p_0 + \varepsilon) \), \( r \in (r - \eta, r + \eta) \), \( t_1 = t_1^* \), and \( \tau = \tau_0 \) for \( \eta \) sufficiently small. Therefore, the inverse cutoff function \( p(c, r, \tau_0, t_1) \) will be identified at \( t_1 = t_1^* \), and \( \tau = \tau_0 \) in a ball centered at \( c(p_0, r^*, \tau_0, t_1^*) \). This implies that \( p_r(c, r, \tau_0, t_1) \) and \( p_c(c, r, \tau_0, t_1) \) are identified over that same region, as are \( \Phi(c, r, \tau_0, t_1) \) and \( \Phi_r(c, r, \tau_0, t_1) \). We have

\[
\frac{U''(c - r)}{U'(c - r)} = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)}.
\]

(SA.6)
Hence, the Arrow–Pratt measure of risk aversion \( \frac{U''}{U'} \) is identified at the point \( c(p_0, r^*, \tau_0, t_1^*) - r^* \). Again, by the same arguments as Section 3.4, \( \delta(\cdot) \) is identified at \( \tau_0 \).

**Appendix SB: Proof that \( U''' \leq 0 \) is a sufficient condition for Assumption 3**

We have

\[
M(v, r, \tau, t) = U^{-1}\left( \delta(\tau) \left( \alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)\,dy \right) \right)
\]
so

\[ M_{v}(v, r, \tau, t) = U^{-1}(\delta(\tau)\left(\alpha(r, \tau, t)U(v-r) + \int_{r}^{v} U(v-y)h(y, \tau, t) \, dy\right)) \]

\[ \cdot \delta(\tau)\left(\alpha(r, \tau, t)U'(v-r) + \int_{r}^{v} U'(v-y)h(y, \tau, t) \, dy\right) \]

\[ = \frac{\delta(\tau)\left(\alpha(r, \tau, t)U'(v-r) + \int_{r}^{v} U'(v-y)h(y, \tau, t) \, dy\right)}{U'(\delta(\tau)\left(\alpha(r, \tau, t)U(v-r) + \int_{r}^{v} U(v-y)h(y, \tau, t) \, dy\right))} \]

\[ < \left(\delta(\tau)\left(\alpha(\cdot)U'(v-r) + \int_{r}^{v} U'(v-y)h(\cdot) \, dy\right)
\]

\[ + \left(1 - \alpha(\cdot) - \int_{r}^{v} h(\cdot) \, dy\right)U'(0)\right) \]

\[ \left/U'(\delta(\tau)\left(\alpha(\cdot)U(v-r) + \int_{r}^{v} U(v-y)h(\cdot) \, dy\right)
\]

\[ + \left(1 - \alpha(\cdot) - \int_{r}^{v} h(\cdot) \, dy\right)U(0)\right)\).\]

The strict inequality holds because of our normalizations that \( U(0) = 0 \) and \( U'(0) = 1 \), and because \( 1 - \alpha(r, \tau, t) - \int_{r}^{v} h(y, \tau, t) \, dy > 0 \) for any finite \( v \).

Therefore, we have

\[ M_{v}(v, r, \tau, t) < \frac{\delta(\tau)EU'(x)}{U'(\delta(\tau)EU(x))}, \]

where the random variable \( x \) has a mixed discrete-continuous distribution, taking the value 0 with probability \( 1 - \alpha(r, \tau, t) - \int_{r}^{v} h(y, \tau, t) \, dy \) and the value \( v-r \) with probability \( \alpha(r, \tau, t) \), and having density \( h(y, \tau, t) \) over the interval \((0, v-r)\). Because \( U'' \leq 0 \) and \( \delta(\tau) < 1 \), Jensen's inequality implies that \( \delta(\tau)EU(x) < U(Ex) \). Therefore,

\[ M_{v}(v, r, \tau, t) < \frac{\delta(\tau)EU'(x)}{U'(U^{-1}(\delta(\tau)EU(x)))} = \frac{\delta(\tau)EU'(x)}{U'(Ex)}. \]

Since \( U''' \leq 0 \), Jensen's inequality implies \( EU'(x) \leq U'(Ex) \). Hence,

\[ M_{v}(v, r, \tau, t) < \delta(\tau) < 1. \]

**Appendix SC: Identification of utility functions from certainty equivalents**

Suppose that \( U \) is a utility function defined on \( \mathcal{X} \subset \mathbb{R} \) and that \( \mathcal{F} \) is a collection of distributions with supports contained in \( \mathcal{X} \). This generates a certainty equivalent functional
(also called a quasilinear mean)

\[ m(F) = U^{-1}\left(\int U(x) \, dF(x)\right), \quad F \in \mathcal{F}. \]

Now suppose that we are given a collection of lotteries \( \mathcal{F} \) and a quasilinear mean functional \( m \). If \( \mathcal{F} \) is sufficiently rich, it is plausible that the utility function \( U \) is uniquely determined (up to affine transformations) by \( m \). We show that this is true even for a well chosen one-dimensional family of lotteries.

Our example is adapted from the proof of Theorem 83 in Hardy, Littlewood, and Polya (1952). Let \( \mathcal{X} = [a, b] \) and consider the collection of lotteries \( \mathcal{F} = \{F_t(x), \, t \in [0, 1]\} \), where the \( F_t \) are mixtures of point masses at the endpoints \( a \) and \( b \):

\[ F_t(x) = (1 - t)\delta_a(x) + t\delta_b(x). \]

Note that

\[ m(F_0) = m(\delta_a) = a, \]
\[ m(F_1) = m(\delta_b) = b, \]

and since \( m \) is continuous and strictly increasing, \( m(F_t) \) takes every value in \([a, b]\).

Suppose that there is another function \( V \) satisfying

\[ m(F) = V^{-1}\left(\int V(x) \, dF(x)\right), \quad F \in \mathcal{F}. \]

Let

\[ \tilde{x}(t) = m(F_t) = U^{-1}\left((1 - t)U(a) + tU(b)\right) = V^{-1}\left((1 - t)V(a) + tV(b)\right). \]

We have

\[ U(\tilde{x}(t)) = (1 - t)U(a) + tU(b), \]

and we can solve for \( t \) and \((1 - t)\):

\[ t = \frac{U(\tilde{x}(t)) - U(a)}{U(b) - U(a)}, \]

\[ (1 - t) = \frac{U(b) - U(\tilde{x}(t))}{U(b) - U(a)}. \]

Now

\[ V(\tilde{x}(t)) = (1 - t)V(a) + tV(b) \]

\[ = \frac{U(b) - U(\tilde{x}(t))}{U(b) - U(a)} \cdot V(a) + \frac{U(\tilde{x}(t)) - U(a)}{U(b) - U(a)} \cdot V(b). \]

This is a linear (in fact, affine) function of \( U(\tilde{x}(t)) \), so we can write

\[ V(\tilde{x}(t)) = \alpha + \beta U(\tilde{x}(t)), \]
where $\alpha$ and $\beta$ do not depend on $t$ and $\beta > 0$. Since this holds for all $t \in [0, 1]$, we have

$$V(x) = \alpha + \beta U(x) \quad \forall x \in [a, b].$$

Thus $V$ must be an affine transformation of $U$.

**References**


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