Online Appendix (not for publication): The Extended Perturbation Method: With Applications to the New Keynesian Model and the Zero Lower Bound

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1 The Certainty Equivalent Solution

This section describes the certainty equivalent solution to DSGE models and how it is obtained numerically by the Extended Path. Section 1.1 introduces our notation and describes the computational problem. We describe how to solve for the certainty equivalent solution in Section 1.2. Good starting values to compute the certainty equivalent solution and how to set the termination values for the control variables are presented in Section 1.3.

1.1 The Setup

The setup is similar to the one given in the paper but included for completeness. That is, we consider the broad class of DSGE models which can be expressed as

$$E_t\left[\mathbf{f}\left(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right)\right] = \mathbf{0} \text{ for all } t = 1, 2, ..., \tag{1}$$

where E_t denotes the conditional expectation given information available in time period t. The state vector $\mathbf{x}_t$ with dimension $n_x \times 1$ belongs to the set $\mathcal{X}_x$, denoting Borel subsets of $\mathbb{R}^{n_x}$. The control variables are stored in $\mathbf{y}_t$ with dimension $n_y \times 1$ and $\mathbf{y}_t \in \mathcal{X}_y$, where $\mathcal{X}_y$ refers to Borel subsets of $\mathbb{R}^{n_y}$. We further let $n_x + n_y = n$. The function $\mathbf{f}$ takes elements from $\mathcal{X}_x \times \mathcal{X}_x \times \mathcal{X}_y \times \mathcal{X}_y$ into $\mathbb{R}^n$.

It is useful to consider the partitioning $\mathbf{x}_t \equiv \left[\begin{array}{cc} \mathbf{x}'_{1,t} & \mathbf{x}'_{2,t} \end{array}\right]'$, where $\mathbf{x}_{1,t}$ contains endogenous state variables and $\mathbf{x}_{2,t}$ denotes exogenous states. The dimensions of these vectors are $n_{x_1} \times 1$ and $n_{x_2} \times 1$, respectively, with $n_{x_1} + n_{x_2} = n_x$. We further assume that the dynamics of the exogenous state variables belong to the general class

$$\mathbf{x}_{2,t+1} = \mathbf{\Gamma}\left(\mathbf{x}_{2,t}\right) + \sigma\bar{\boldsymbol{\eta}}\boldsymbol{\epsilon}_{t+1}, \tag{2}$$

where $\boldsymbol{\epsilon}_{t+1} \in \mathcal{X}_\epsilon$ and has dimension $n_\epsilon \times 1$. We also assume $\boldsymbol{\epsilon}_{t+1}$ to be independent and identically distributed with zero and an unit co-variance matrix, i.e. $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID}\left(\mathbf{0}, \mathbf{I}\right)$. The function $\mathbf{\Gamma}$ maps elements from $\mathcal{X}_{x_2}$ into $\mathcal{X}_{x_2}$. We further assume that $\mathbf{\Gamma}$ generates a stable process for $\mathbf{x}_{2,t}$.[1] In linear systems, this corresponds to requiring that all eigenvalues of the $n_{x_2} \times n_{x_2}$ Jacobian $\partial\mathbf{\Gamma}/\partial\mathbf{x}'_{2,t}$ lie inside the unit circle. For non-linear mappings, $\mathbf{\Gamma}$ must satisfy the general stability condition for nonlinear first-order Markov systems provided in the paper.

In the certainty equivalent solution, uncertainty about future shocks is absence by definition. This corresponds to omitting the conditional expectation in (1), which reduces to a deterministic system, i.e.

$$\mathbf{f}\left(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) = \mathbf{0} \text{ for all } t = 1, 2, ...$$

The system contains an infinite number of equations and cannot be solved without any simplifying assumptions. The approach taken in the Extended Path of Fair and Taylor (1983) is to truncate the problem at some finite horizon N after which the variables are assumed to be constant - for instance, given by their deterministic steady state values. Although this is an asymptotic model property it implies that the approximation errors from the truncation decreases in N and can be made arbitrary small for an appropriately chosen horizon N (see Fair and Taylor (1983) and Boucekkine (1995)). Hence, the key assumption in the Extended Path is to close the infinite system in (1) by considering a terminal value for $\mathbf{y}_{t+N}$. This gives rise to the finite dimensional system

$$\begin{aligned}
\mathbf{f}\left(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) &= \mathbf{0}_{n\times 1} \\
\mathbf{f}\left(\mathbf{x}_{t+1}, \mathbf{x}_{t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}\right) &= \mathbf{0}_{n\times 1} \\
\mathbf{f}\left(\mathbf{x}_{t+2}, \mathbf{x}_{t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}\right) &= \mathbf{0}_{n\times 1} \\
&\cdots \\
\mathbf{f}\left(\mathbf{x}_{t+N-1}, \mathbf{x}_{t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}\right) &= \mathbf{0}_{n\times 1}.
\end{aligned} \tag{3}$$

Given that $\mathbf{f}\left(\cdot\right)$ containts n equilibrium conditions, we thus have a total of $n * N$ conditions. The initial state $\mathbf{x}_t$ and $\mathbf{y}_{t+N}$ are known by assumption, whereas we must solve for $\left(\mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{y}_{t+1}, \mathbf{x}_{t+2}, \mathbf{y}_{t+2}, ..., \mathbf{x}_{t+N-1}, \mathbf{y}_{t+N-1}, \mathbf{x}_{t+N}\right)$, consituting $n*N$ unknowns. Thus, the certainty equivalent solution as given by the Extended Path is obtained by solving the fixed-point problem implied by (3).

[1] This implies that trends may only be included in the class of DSGE models considered, if a given model after re-scaling has an equivalent representation without trending variables. The procedure is carefully described in King and Rebelo (1999).

1.1.1 System Reduction I

To reduce the computational burden when solving the fixed-point problem in (3), the exogenous states $\{\mathbf{x}_{2,t+j}\}_{j=1}^{N}$ can be concentrated out of the system as they can be computed directly by iterating on (2). Hence, the concentrated fixed-point problem is given by

$$\begin{aligned} \mathbf{f}_1\left(\mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) &= \mathbf{0}_{n_1\times 1} \\ \mathbf{f}_1\left(\mathbf{x}_{1,t+1}, \mathbf{x}_{1,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}\right) &= \mathbf{0}_{n_1\times 1} \\ \mathbf{f}_1\left(\mathbf{x}_{1,t+2}, \mathbf{x}_{1,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}\right) &= \mathbf{0}_{n_1\times 1} \\ &\dots \\ \mathbf{f}_1\left(\mathbf{x}_{1,t+N-1}, \mathbf{x}_{1,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}\right) &= \mathbf{0}_{n_1\times 1}, \end{aligned} \tag{4}$$

where we have introduced the partitioning

$$\mathbf{f}\left(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) \equiv \begin{bmatrix} \mathbf{f}_1\left(\mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) \\ \mathbf{f}_2\left(\mathbf{x}_{2,t}, \mathbf{x}_{2,t+1}\right) \end{bmatrix}.$$

The function $\mathbf{f}_2(\cdot)$ has dimensions $n_2 \times 1$ where $n_2 \equiv n_{x_2}$ and contains the law of motions for the exogenous variables, whereas the function $\mathbf{f}_1(\cdot)$ has dimensions $n_1 \times 1$ where $n_1 \equiv n_y + n_{x_1}$ and contains all the remaining equilibrium conditions. In $\mathbf{f}_1(\cdot)$, we suppress the dependence of $\mathbf{x}_{2,t}$ to reduce the notational burden. Hence, the concentrated system in (4) has $n_1 * N$ equations with the same number of unknowns and constitutes the certainty equivalent solution, provided that it is unaffected by increasing N (see Fair and Taylor (1983)). For numerical stability of the proposed routines below, we recommend expressing the residuals of $\mathbf{f}$ in terms of unit-free errors.

1.1.2 System Reduction II

Some of the most popular DSGE models are characterized by having lagged control variables in the states. For instance, lagged consumption c_{t-1} to capture the effect of consumption habits in the RBC model or the lagged interest rate r_{t-1} to capture interest rate smoothing by the central bank in a New Keynesian model. This link between some of the controls and the states can be exploited to futher eliminate some of the states, as we do not need to separately solve for a path of $\{c_t\}_{t=1}^{N}$ and $\{c_{t-1}\}_{t=1}^{N}$, say. The same observation is used in Binning (2013) to reduce the number of states in the standard perturbation approximation. To see how we concentrate these lagged control variables out of the system, consider the following partion

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{x}_{11,t} \\ \mathbf{x}_{12,t} \\ \mathbf{x}_{2,t} \end{bmatrix}.$$

Here, $\mathbf{x}_{11,t}$ contains the true endogenous state variables (such as capital), whereas $\mathbf{x}_{12,t}$ contains the lagged control variables (which also are endogenous states). As in Binning (2013), we say that $\mathbf{x}_{11,t}$ has dimension $m_x \times 1$ and $\mathbf{x}_{12,t}$ has dimension $m_{yx} \times 1$. As before, $\mathbf{x}_{2,t}$ refers to the exogenous states. Hence, the concentrated fixed-point problem is given by

$$\begin{aligned} \mathbf{f}_1\left(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) &= \mathbf{0}_{n_1\times 1} \\ \mathbf{f}_1\left(\mathbf{y}_{t}, \mathbf{x}_{11,t+1}, \mathbf{x}_{11,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}\right) &= \mathbf{0}_{n_1\times 1} \\ \mathbf{f}_1\left(\mathbf{y}_{t+1}, \mathbf{x}_{11,t+2}, \mathbf{x}_{11,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}\right) &= \mathbf{0}_{n_1\times 1} \\ &\dots \\ \mathbf{f}_1\left(\mathbf{y}_{t+N-2}, \mathbf{x}_{11,t+N-1}, \mathbf{x}_{11,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}\right) &= \mathbf{0}_{n_1\times 1}, \end{aligned} \tag{5}$$

where we have introduced the partitioning

$$\mathbf{f}\left(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) \equiv \begin{bmatrix} \mathbf{f}_1\left(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) \\ \mathbf{f}_{12}\left(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}\right) \\ \mathbf{f}_2\left(\mathbf{x}_{2,t}, \mathbf{x}_{2,t+1}\right) \end{bmatrix}.$$

Note that we here explicitely account for the fact that $\mathbf{y}_{t-1}$ enters in the $\mathbf{f}$-function, as this effect is not captured by $\mathbf{x}_{11,t}$. This is what makes this additional concentration "trick" different from the one presented in Section 1.1.1. Furthermore, $\mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1})$ contains the first $n_y + m_x$ equations in the model, $\mathbf{f}_{12}(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1})$ contains the m_{yx} link-equations for defining the lagged control variables, and finally $\mathbf{f}_2(\cdot)$ contains the exogenous shocks with dimension $n_2 \times 1$ where $n_2 \equiv n_{x_2}$.

1.2 Solving the Fixed-Point Problem

This subsection describes how we solve for the fixed-point problem in (4) when using the system reduction described in Section 1.1.1. To do so, we let the unknowns be denoted by

$$\mathbf{Z} \equiv \begin{bmatrix} \mathbf{y}_t & \mathbf{y}_{t+1} & \cdots & \mathbf{y}_{t+N-2} & \mathbf{y}_{t+N-1} \\ \mathbf{x}_{1,t+1} & \mathbf{x}_{1,t+2} & \cdots & \mathbf{x}_{1,t+N-1} & \mathbf{x}_{1,t+N} \end{bmatrix},$$

meaning that the system in (4) can be condensely expressed as

$$\mathbf{F}(\mathbf{Z}) \equiv \begin{bmatrix} \mathbf{f}_1(\mathbf{x}_{1,t}, \mathbf{x}_{1,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_1(\mathbf{x}_{1,t+1}, \mathbf{x}_{1,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) \\ \mathbf{f}_1(\mathbf{x}_{1,t+2}, \mathbf{x}_{1,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) \\ \cdots \\ \mathbf{f}_1(\mathbf{x}_{1,t+N-1}, \mathbf{x}_{1,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) \end{bmatrix} = \mathbf{0}.$$

Linearizing this system around the point $\mathbf{Z}^*$ gives

$$\mathbf{F}(\mathbf{Z}) \approx \mathbf{F}(\mathbf{Z}^*) + \mathbf{J}(\mathbf{Z}^*)(vec(\mathbf{Z}) - vec(\mathbf{Z}^*)), \tag{6}$$

where $\mathbf{J}(\mathbf{Z}^*)$ denotes the Jacobian evaluated at $\mathbf{Z}^*$. That is,

$$\mathbf{J}(\mathbf{Z}^*) \equiv \left. \frac{\partial \mathbf{F}(\mathbf{Z})}{\partial vec(\mathbf{Z})'} \right|_{\mathbf{Z}=\mathbf{Z}^*},$$

which has dimensions $(n_1 * N) \times (n_1 * N)$.

When also using the system reduction in Section 1.1.2, we define

$$\mathbf{Z} \equiv \begin{bmatrix} \mathbf{y}_t & \mathbf{y}_{t+1} & \cdots & \mathbf{y}_{t+N-2} & \mathbf{y}_{t+N-1} \\ \mathbf{x}_{11,t+1} & \mathbf{x}_{11,t+2} & \cdots & \mathbf{x}_{11,t+N-1} & \mathbf{x}_{11,t+N} \end{bmatrix}$$

and the system in (5) can be condensely expressed as

$$\mathbf{F}(\mathbf{Z}) \equiv \begin{bmatrix} \mathbf{f}_1(\mathbf{x}_{11,t}, \mathbf{x}_{11,t+1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \mathbf{f}_1(\mathbf{x}_{11,t+1}, \mathbf{x}_{11,t+2}, \mathbf{y}_{t+1}, \mathbf{y}_{t+2}) \\ \mathbf{f}_1(\mathbf{x}_{11,t+2}, \mathbf{x}_{11,t+3}, \mathbf{y}_{t+2}, \mathbf{y}_{t+3}) \\ \cdots \\ \mathbf{f}_1(\mathbf{x}_{11,t+N-1}, \mathbf{x}_{11,t+N}, \mathbf{y}_{t+N-1}, \mathbf{y}_{t+N}) \end{bmatrix} = \mathbf{0}.$$

1.2.1 The Standard Newton-Raphson Routine

The most efficeint way to solve (4) (if we can obtain convergence) is to use the Newton-Raphson routine. To describe this routine, let $\mathbf{Z}^i$ denote the value of $\mathbf{Z}$ at iteration i, and use (6) for $\mathbf{Z}^i = \mathbf{Z}^*$ to obtain

$$\mathbf{F}(\mathbf{Z}) \approx \mathbf{F}(\mathbf{Z}^i) + \mathbf{J}(\mathbf{Z}^i)(vec(\mathbf{Z}) - vec(\mathbf{Z}^i)).$$

Let $\mathbf{F}(\mathbf{Z}) = \mathbf{0}$ and isolate for $vec(\mathbf{Z})$, i.e.

$$vec(\mathbf{Z}) = vec(\mathbf{Z}^i) - \mathbf{J}(\mathbf{Z}^i)^{-1}\mathbf{F}(\mathbf{Z}^i).$$

Hence, the Newton-Raphson routine is then

$$vec\left(\mathbf{Z}^{i+1}\right) = vec\left(\mathbf{Z}^{i}\right) - \mathbf{J}\left(\mathbf{Z}^{i}\right)^{-1}\mathbf{F}\left(\mathbf{Z}^{i}\right). \tag{7}$$

We iterate until the change in max $\left|vec\left(\mathbf{Z}^{i+1}\right) - vec\left(\mathbf{Z}^{i}\right)\right|$ is smaller than some tolerance. Note that this routine is sometimes referred to as a Newton-Raphson relaxation algorithm (see Boucekkine (1995)). When computing the inversion of the Jacobian we use the efficient method of Boucekkine (1995) (see Section 2 of this technical appendix for a presentation of this method using our notation).

1.2.2 The Extended Newton-Raphson routine

When the problem is very nonlinear, the standard Newton-Raphson may fail to converge. If this is the case, then we consider an extended Newton-Raphson routine of the form

$$vec\left(\mathbf{Z}^{i+1}\left(\delta\right)\right) = vec\left(\mathbf{Z}^{i}\right) - \delta\left(\mathbf{J}\left(\mathbf{Z}^{i}\right)\right)^{-1}\mathbf{F}\left(\mathbf{Z}^{i}\right), \tag{8}$$

where the scaling parameter $\delta \in \mathbb{R}$ is determined by a rough grid search to the problem

$$\delta = \arg\min_{\delta} \quad \mathbf{F}\left(\mathbf{Z}^{i+1}\left(\delta\right)\right)'\mathbf{F}\left(\mathbf{Z}^{i+1}\left(\delta\right)\right).$$

That is, δ accounts for the possibility that a linear approximation to $\mathbf{F}(\mathbf{Z})$ may be insufficiently accurate. We iterate on (8) until the change in max $\left|vec\left(\mathbf{Z}^{i+1}\right) - vec\left(\mathbf{Z}^{i}\right)\right|$ is smaller than some tolerance, and we refer to this algorithm as an extended Newton-Raphson relaxation routine.

1.2.3 Minimizing the Squared Model-Residuals by the LM Optimizer

If these routines are unsuccesful in solving the fixed-point problem in (4), then we use the Levenberg-Marquardt (LM) optimizer to minimize $\mathbf{F}(\mathbf{Z})'\mathbf{F}(\mathbf{Z})$ across $\mathbf{Z}$.

1.3 Determining Starting Values and Terminal Values for the Controls

To obtain fast convergence when solving the fixed-point problem in (4) it is essiential to have good starting values. They are typically computed based on a first-order approximation, i.e. by iterating on

$$\mathbf{y}_{t+i}^{1st} - \mathbf{y}_{ss} = \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_{t+i}^{1st} - \mathbf{x}_{ss}\right) \tag{9}$$

$$\mathbf{x}_{t+i}^{1st} - \mathbf{x}_{ss} = \mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_{t+i-1}^{1st} - \mathbf{x}_{ss}\right)$$

for $i = 1, 2, ..., N$ where $\mathbf{x}_{t}^{1st} = \mathbf{x}_{t}$. Adjemian and Juillard (2010) further notes that it is possible to reduce the value of N and hence increase the speed of the certainty equivalent solver by replacing the standard terminal condition of $\mathbf{y}_{t+N} = \mathbf{y}_{ss}$ by $\mathbf{y}_{t+N} = \mathbf{y}_{t+N}^{1st}$. Of course, for $N \to \infty$, we clearly have $\mathbf{x}_{t+i}^{1st} - \mathbf{x}_{ss} \longrightarrow \mathbf{0}$ and $\mathbf{y}_{t+i}^{1st} - \mathbf{y}_{ss} \longrightarrow \mathbf{0},$ given the stability of $\mathbf{h}_{\mathbf{x}}$, implying that this procedure reproduces the standard terminal condition in the limit. This section generalizes these ideas by using a fourth-order perturbation approximation to compute starting values and the terminal value of $\mathbf{y}_{t+N}$. To reduce the computational burden, we use the perturbation on perturbation (POP) method suggested by Andreasen and Zabczyk (2015) and applied to compute conditional expectations in the technical appendix of Andreasen (2012). Below, we provide the key results from this technical appendix for terms under certainty equivalence (i.e. we leave out the expressions for derivatives involving the perturbation parameter) and extend these results to fourth order.

1.3.1 Using the POP-Method to Compute Conditional Expectations in DSGE models

Consider the case where the DSGE model reports the endogenous variable r_t and we want to compute conditional expectations of this variable, i.e. $r_{1,t} \equiv E_t[r_{t+1}]$, $r_{2,t} \equiv E_t[r_{t+2}]$, $r_{3,t} \equiv E_t[r_{t+3}]$, etc.[2] The law of iterated expectations implies $r_{2,t} \equiv E_t[r_{t+2}] = E_t[E_{t+1}[r_{t+2}]] = E_t[r_{1,t+1}]$ and so on. Hence, we only need to derive a formula for computing $p_t \equiv E_t[r_{t+1}]$ because all other expectations can be found be iterating this formula. We therefore consider the problem

$$p(\mathbf{x}_t, \sigma) = E_t[r(\mathbf{x}_{t+1}, \sigma)] \tag{10}$$

where σ is the perturbation parameter. We then observe that

$$F(\mathbf{x}_t, \sigma) \equiv E_t\left[-p(\mathbf{x}_t, \sigma) + r\left(\mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}, \sigma\right)\right] = 0, \tag{11}$$

because

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}. \tag{12}$$

Note then that (11) must hold for all values of $(\mathbf{x}_t, \sigma)$. This allow us to compute all derivatives of p with respect to $(\mathbf{x}_t, \sigma)$ around the deterministic steady state, i.e. $\mathbf{x}_t = \mathbf{x}_{ss}$ and $\sigma = 0$, given derivatives of $\mathbf{h}(\mathbf{x}_t, \sigma)$ and $r(\mathbf{x}_{t+1}, \sigma)$ around the same point.

For the indices we adopt the convention that the subscript indicates the order of differentiation. I.e. a subscript 1 is for the first time we take derivatives and so on. Thus,

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, 2, ...n_x$$

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 = 1, 2, ...n_x$$

where the $\alpha's$ and $\gamma's$ will be used to index the states. We will also use superscript t and $t+1$ on p, r, and $\mathbf{h}$ and their derivatives. This is done to indicate that the functions are functions of $\mathbf{x}_t$ and $\mathbf{x}_{t+1}$, respectively. This distinction is relevant when we compute the derivatives, but not when these derivatives are evaluated in the deterministic steady state where $\mathbf{x}_t = \mathbf{x}_{t+1} = \mathbf{x}_{ss}$.

1.3.2 The First Order Terms: For $(\mathbf{x}_t)$

$$[F_{\mathbf{x}}(\mathbf{x}_{ss}, 0)]_{\alpha_1} = E_t\left[-\left[p^t_{\mathbf{x}}\right]_{\alpha_1} + \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\right] = 0$$

$\Updownarrow$

$$\left[p^t_{\mathbf{x}}\right]_{\alpha_1} = E_t\left[\left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}\right].$$

Hence, in the steady state we have

$$[p_{\mathbf{x}}]_{\alpha_1} = [r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1},$$

or in standard matrix notation

$$\mathbf{p}_{\mathbf{x}}(1;:) = \mathbf{r}_{\mathbf{x}}(1,:)\,\mathbf{h}_{\mathbf{x}},$$

which is identical to the expression computed in (9).

1.3.3 The Second Order Terms: For $(\mathbf{x}_t, \mathbf{x}_t)$

$$[F_{\mathbf{xx}}(\mathbf{x}_{ss}, 0)]_{\alpha_1\alpha_2} = E_t[-\left[p^t_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} + \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}] = 0$$

$\Updownarrow$

$$\left[p^t_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} = E_t\left[\left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} + \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}\right].$$

Hence, in the steady state we have

$$[p_{\mathbf{xx}}]_{\alpha_1\alpha_2} = [r_{\mathbf{xx}}]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} + [r_{\mathbf{x}}]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2},$$

or in the standard matrix notation

$$\mathbf{p}_{\mathbf{xx}} = \mathbf{h}'_{\mathbf{x}}\mathbf{r}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}} + \textstyle\sum_{\gamma_1=1}^{n_x}\mathbf{r}_{\mathbf{x}}(1, \gamma_1)\,\mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :).$$

[2] If the variable of interest is a control variable, then the function $r(\mathbf{x}_{t+1}, \sigma)$ follows from the $\mathbf{g}$-function. If the variable of interest is a state variable, then we let $r_t \equiv \mathbf{i}'\mathbf{x}_t$ to obtain moments for the i'th state variable with $i(k,1) = 1$ for $k = i$, otherwise $i(k,1) = 0$.

1.3.4 Third Order Terms: For $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$

$$\begin{aligned}
\left[F_{\mathbf{xxx}}\left(\mathbf{x}_{ss},0\right)\right]_{\alpha_1\alpha_2\alpha_3} &= E_t[-\left[p^t_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} \\
&+ \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
&+ \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} \\
&= 0
\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}
\left[p^t_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} &= E_t[\left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
&+ \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}].
\end{aligned}$$

Hence, in the steady state we have

$$\begin{aligned}
\left[p_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} &= \left[r_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+ \left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+ \left[r_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\
&+ \left[r_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3},
\end{aligned}$$

or in the standard matrix notation

$$\begin{aligned}
\mathbf{p}_{\mathbf{xxx}}\left(\alpha_1,\alpha_2,\alpha_3\right) &= \textstyle\sum_{\gamma_3=1}^{n_x}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_1\right)'\mathbf{r}_{\mathbf{xxx}}\left(:,:,\gamma_3\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_2\right)\mathbf{h}_{\mathbf{x}}\left(\gamma_3,\alpha_3\right) \\
&+\mathbf{h}_{\mathbf{x}}\left(:,\alpha_1\right)'\mathbf{r}_{\mathbf{xx}}\mathbf{h}_{\mathbf{xx}}\left(:,\alpha_2,\alpha_3\right) \\
&+\textstyle\sum_{\gamma_1=1}^{n_x}\mathbf{r}_{\mathbf{xx}}\left(\gamma_1,:\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_2\right)\mathbf{h}_{\mathbf{xx}}\left(\gamma_1,\alpha_1,\alpha_3\right) \\
&+\textstyle\sum_{\gamma_1=1}^{n_x}\mathbf{r}_{\mathbf{xx}}\left(\gamma_1,:\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)\mathbf{h}_{\mathbf{xx}}\left(\gamma_1,\alpha_1,\alpha_2\right) \\
&+\mathbf{r}_{\mathbf{x}}\left(1,:\right)\mathbf{h}_{\mathbf{xxx}}\left(:,\alpha_1,\alpha_2,\alpha_3\right).
\end{aligned}$$

1.3.5 Fourth Order Terms: For $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$

Recall that

$$\begin{aligned}[F_{\mathbf{xxx}}(\mathbf{x}_{ss},0)]_{\alpha_1\alpha_2\alpha_3} &= E_t[-\left[p^t_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} \\ &+ \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}^t_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \\ &+ \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} \\ &= 0\end{aligned}$$

$$[F_{\mathbf{xxx}}(\mathbf{x}_{ss},0)]_{\alpha_1\alpha_2\alpha_3} = E_t[-[p^t_{\mathbf{xxx}}]_{\alpha_1\alpha_2\alpha_3} + Q_1 + Q_2 + Q_3 + Q_4 + Q_5]$$

where

$Q_1 = \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$Q_2 = \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

$Q_3 = \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$

$Q_4 = \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$

$Q_5 = \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$

Thus,

$$\begin{aligned}[Q_{1,\mathbf{x}}]_{\alpha_4} = &\left[r^{t+1}_{4\mathbf{x}}\right]_{\gamma_1\gamma_2\gamma_3\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_3}_{\alpha_3\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_4}\end{aligned}$$

$$\begin{aligned}[Q_{2,\mathbf{x}}]_{\alpha_4} = &\left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_2}_{\alpha_2\alpha_3\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_4}\end{aligned}$$

$$\begin{aligned}[Q_{3,\mathbf{x}}]_{\alpha_4} = &\left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_4}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_3\alpha_4}\end{aligned}$$

$$\begin{aligned}[Q_{4,\mathbf{x}}]_{\alpha_4} = &\left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_3\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_3}_{\alpha_3\alpha_4}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2} \\ &+ \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_4}\end{aligned}$$

$$[Q_{5,\mathbf{x}}]_{\alpha_4} = \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} + \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}^t_{\mathbf{xxxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3\alpha_4}$$

Thus

$[F_{\mathbf{4x}}(\mathbf{x}_{ss},0)]_{\alpha_1\alpha_2\alpha_3\alpha_4} = E_t[-[p^t_{\mathbf{4x}}]_{\alpha_1\alpha_2\alpha_3\alpha_4}$

1) $\quad + \left[r^{t+1}_{\mathbf{4x}}\right]_{\gamma_1\gamma_2\gamma_3\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

2) $\quad + \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_3}_{\alpha_3\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

3) $\quad + \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

4) $\quad + \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_4}$

5) $\quad + \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

6) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_2}_{\alpha_2\alpha_3\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$

7) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_4}$

8) $\quad + \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$

9) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_2}_{\alpha_2\alpha_4}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_3}$

10) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_2}_{\alpha_2}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_3\alpha_4}$

11) $\quad + \left[r^{t+1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_3\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$

12) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_3}_{\alpha_3\alpha_4}[\mathbf{h}^t_{\mathbf{xx}}]^{\gamma_1}_{\alpha_1\alpha_2}$

13) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_4}$

14) $\quad + \left[r^{t+1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_4}[\mathbf{h}^t_{\mathbf{x}}]^{\gamma_4}_{\alpha_4}[\mathbf{h}^t_{\mathbf{xxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$

15) $\quad + \left[r^{t+1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}^t_{\mathbf{xxxx}}]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3\alpha_4}]$

$= 0$

2 Efficient Inversion of the Jacobian: System Reduction I

This section presents the efficient solution algorithm of Boucekkine (1995) to compute $\mathbf{J}(\mathbf{Z})^{-1}\mathbf{F}(\mathbf{Z})$. In this section, we consider

$$\mathbf{f}(\mathbf{x}_t,\mathbf{y}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1}) = \mathbf{0}$$

or the condense version of this system, i.e. $\mathbf{f}_1(\mathbf{x}_{1,t},\mathbf{y}_t,\mathbf{x}_{1,t+1},\mathbf{y}_{t+1}) = \mathbf{0}$. Whether we use the normal or condense system in Section 1.1.1, we will express it as

$$\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{xp},\mathbf{yp}) = \mathbf{0}.$$

To reduce the notational burden let

$$\mathbf{r}_t \equiv [\mathbf{x}_t,\mathbf{y}_t,\mathbf{x}_{t+1},\mathbf{y}_{t+1}],$$

or

$$\mathbf{r}_t \equiv [\mathbf{x}_{1,t},\mathbf{y}_t,\mathbf{x}_{1,t+1},\mathbf{y}_{t+1}],$$

if using the condensed system. We then consider the following notation for the partial derivatives

$$
\begin{aligned}
\mathbf{f_x}\left(\mathbf{r}_t\right) &= \frac{\partial \mathbf{f}\left(\mathbf{r}_t\right)}{\partial \mathbf{x}} \\
\mathbf{f_y}\left(\mathbf{r}_t\right) &= \frac{\partial \mathbf{f}\left(\mathbf{r}_t\right)}{\partial \mathbf{y}} \\
\mathbf{f_{xp}}\left(\mathbf{r}_t\right) &= \frac{\partial \mathbf{f}\left(\mathbf{r}_t\right)}{\partial \mathbf{xp}} \\
\mathbf{f_{yp}}\left(\mathbf{r}_t\right) &= \frac{\partial \mathbf{f}\left(\mathbf{r}_t\right)}{\partial \mathbf{yp}}.
\end{aligned}
$$

To understand the logic of the solution method of Boucekkine (1995) it is useful to gradually extend the horizon N when solving the certainty equivalent solution.

2.1 2 Time Periods

Let $N = 2$ and consider the system

$$
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{f_x}(\mathbf{r}_t) & \mathbf{f_y}(\mathbf{r}_t) & \mathbf{f_{xp}}(\mathbf{r}_t) & \mathbf{f_{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f_x}(\mathbf{r}_{t+1}) & \mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2}
\end{bmatrix}
= -
\begin{bmatrix}
-\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ -\mathbf{y}_{t+N}
\end{bmatrix}
$$

$$\Downarrow$$

$$
\underbrace{\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
-\mathbf{f_x}(\mathbf{r}_t) & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n} & \mathbf{I}_{n_y}
\end{bmatrix}}_{3n\times 3n}
\underbrace{\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{f_x}(\mathbf{r}_t) & \mathbf{f_y}(\mathbf{r}_t) & \mathbf{f_{xp}}(\mathbf{r}_t) & \mathbf{f_{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f_x}(\mathbf{r}_{t+1}) & \mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}}_{3n\times 3n}
\begin{bmatrix}
\mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
-\mathbf{f_x}(\mathbf{r}_t) & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_t \\ -\mathbf{f}(\mathbf{r}_t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

$$\Updownarrow$$

$$
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{f_x}(\mathbf{r}_t) - \mathbf{f_x}(\mathbf{r}_t) & \mathbf{f_y}(\mathbf{r}_t) & \mathbf{f_{xp}}(\mathbf{r}_t) & \mathbf{f_{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f_x}(\mathbf{r}_{t+1}) & \mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{x}_t \\ -\mathbf{f_x}(\mathbf{r}_t)\,\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

$$\Updownarrow$$

$$
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{f_y}(\mathbf{r}_t) & \mathbf{f_{xp}}(\mathbf{r}_t) & \mathbf{f_{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f_x}(\mathbf{r}_{t+1}) & \mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{x}_t \\ -\mathbf{f_x}(\mathbf{r}_t)\,\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{I}_{n,n_y} & \mathbf{I}_{n,n_x} & \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{W}_t^{-1}\left(-\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t)\right) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

where $\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) \end{bmatrix}$ and we use the partioning $\mathbf{I}_n \equiv \begin{bmatrix} \mathbf{I}_{n,n_y} & \mathbf{I}_{n,n_x} \end{bmatrix}$ where $\mathbf{I}_{n,n_y}$ are the first n_y columns of $\mathbf{I}_n$ and $\mathbf{I}_{n,n_x}$ denotes the last columns of $\mathbf{I}_n$.

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} \equiv \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t)$ and $\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1}\left(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t)\right)$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) - \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

where we have defined $\mathbf{W}_{t+1} \equiv \begin{bmatrix} -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) \end{bmatrix}$ and $\begin{bmatrix} \mathbf{Q}_{n_y}(t+1) \\ \mathbf{Q}_{n_x}(t+1) \end{bmatrix} \equiv \mathbf{W}_{t+1}^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1})$ and $\begin{bmatrix} \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \end{bmatrix} \equiv -\mathbf{W}_{t+1}^{-1}(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1}))$

This system can then be solved by backward substitution as follows:

- $\mathbf{y}_{t+2} = \mathbf{y}_{t+N}$
- $\mathbf{x}_{t+2} = \mathbf{R}_{n_x}(t+1) - \mathbf{Q}_{n_x}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{y}_{t+1} = \mathbf{R}_{n_y}(t+1) - \mathbf{Q}_{n_y}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{x}_{t+1} = \mathbf{R}_{n_x}(t) - \mathbf{Q}_{n_x}(t)\mathbf{y}_{t+1}$
- $\mathbf{y}_t = \mathbf{R}_{n_y}(t) - \mathbf{Q}_{n_y}(t)\mathbf{y}_{t+1}$

Recall that for period t

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} \equiv \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t)$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1}(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t))$$

and for period $t+1$

$$\mathbf{W}_{t+1} \equiv \begin{bmatrix} -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t+1) \\ \mathbf{Q}_{n_x}(t+1) \end{bmatrix} \equiv \mathbf{W}_{t+1}^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1})$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \end{bmatrix} \equiv -\mathbf{W}_{t+1}^{-1}(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1}))$$

2.2 3 Time Periods

Let $N = 3$ and consider the system:

$$\underbrace{\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix}}_{4n\times 4n} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix} = - \begin{bmatrix} -\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{f}(\mathbf{r}_{t+2}) \\ -\mathbf{y}_{t+N} \end{bmatrix}$$

Using the same transformations as above for the two first periods we have

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ -\mathbf{R}_{n_y}(t) \\ -\mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & \mathbf{I}_{n} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & \mathbf{I}_{n} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\,\mathbf{Q}_{n_x}(t+1) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\,\mathbf{R}_{n_x}(t+1) - \mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$$\Updownarrow$$

$$\left[\begin{array}{cccccccc} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{W}_{t+2} & - & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{array}\right] = \left[\begin{array}{c} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\,\mathbf{R}_{n_x}(t+1) - \mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N} \end{array}\right]$$

Let $\mathbf{W}_{t+2} \equiv \left[\begin{array}{cc} -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\,\mathbf{Q}_{n_x}(t+1) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+2}) \end{array}\right]$

$\Updownarrow$

$$\left[\begin{array}{cccccccc} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+2) \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+2) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{array}\right] = \left[\begin{array}{c} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \\ \mathbf{y}_{t+N} \end{array}\right]$$

where $\left[\begin{array}{c} \mathbf{Q}_{n_y}(t+2) \\ \mathbf{Q}_{n_x}(t+2) \end{array}\right] \equiv \mathbf{W}_{t+2}^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2})$ and $\left[\begin{array}{c} \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \end{array}\right] \equiv -\mathbf{W}_{t+2}^{-1}\left(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\,\mathbf{R}_{n_x}(t+1) + \mathbf{f}(\mathbf{r}_{t+2})\right)$.
Thus this system is easily solved by backwards substitution.

2.3 Summarizing: For N time periods

Thus the general solution is given by

- $\mathbf{y}_{t+N} = \mathbf{y}_{t+N}$ is given
- $\mathbf{y}_{t+k} = -\mathbf{R}_{n_y}(t+k) - \mathbf{Q}_{n_y}(t+k)\,\mathbf{y}_{t+k+1}$ for $k = 0, 1, 2, ..., N-1$
- $\mathbf{x}_{t+k+1} = -\mathbf{R}_{n_x}(t+k) - \mathbf{Q}_{n_x}(t+k)\,\mathbf{y}_{t+k+1}$ for $k = 0, 1, 2, ..., N-1$

The relevant matrices are given by:

$$\mathbf{W}_t \equiv \left[\begin{array}{cc} \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{array}\right] \equiv \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t)$$

$$\left[\begin{array}{c} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{array}\right] \equiv -\mathbf{W}_t^{-1}\left(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\,\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t)\right)$$

and for period $t+k$

$$\mathbf{W}_{t+k} \equiv \left[\begin{array}{cc} -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+k})\,\mathbf{Q}_{n_x}(t+k-1) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+k}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+k}) \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{Q}_{n_y}(t+k) \\ \mathbf{Q}_{n_x}(t+k) \end{array}\right] \equiv \mathbf{W}_{t+k}^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+k})$$

$$\left[\begin{array}{c} \mathbf{R}_{n_y}(t+k) \\ \mathbf{R}_{n_x}(t+k) \end{array}\right] \equiv -\mathbf{W}_{t+k}^{-1}\left(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+k})\,\mathbf{R}_{n_x}(t+k-1) + \mathbf{f}(\mathbf{r}_{t+k})\right)$$

for $k = 1, 2, ..., N$

3 Efficient Inversion of the Jacobian: System Reduction II

This section presents the efficient solution algorithm of Boucekkine (1995) to compute $\mathbf{J}(\mathbf{Z})^{-1}\mathbf{F}(\mathbf{Z})$. In this section, we consider

$$\mathbf{f}_1(\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{y}_t, \mathbf{x}_{11,t+1}, \mathbf{y}_{t+1}) = \mathbf{0}$$

To condensely represent this system of equations, we use the notation

$$\mathbf{f}(\mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{xp}, \mathbf{yp}) = \mathbf{0},$$

meaning that $\mathbf{z}$ refers to $\mathbf{y}_{t-1}$, $\mathbf{x}$ to $\mathbf{x}_{11,t}$, $\mathbf{xp}$ to $\mathbf{x}_{11,t+1}$, $\mathbf{y}$ to $\mathbf{y}_t$, and $\mathbf{yp}$ to $\mathbf{y}_{t+1}$. Also, to reduce the notational burden, let

$$\mathbf{r}_t \equiv [\mathbf{y}_{t-1}, \mathbf{x}_{11,t}, \mathbf{y}_t, \mathbf{x}_{11,t+1}, \mathbf{y}_{t+1}].$$

We then consider the following notation for the partial derivatives

$$\begin{aligned}
\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) &= \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{x}} \\
\mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) &= \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{y}} \\
\mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) &= \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{xp}} \\
\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) &= \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{yp}} \\
\mathbf{f}_{\mathbf{z}}(\mathbf{r}_t) &= \frac{\partial \mathbf{f}(\mathbf{r}_t)}{\partial \mathbf{z}}
\end{aligned}$$

To understand the logic of the solution method of Boucekkine (1995) it is useful to gradually extend the horizon N when solving the certainty equivalent solution.

3.1 2 Time Periods

Let $N = 2$ and consider the system

$$\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}
= -\begin{bmatrix} -\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ -\mathbf{y}_{t+N} \end{bmatrix}$$

$\Downarrow$

$$\underbrace{\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
-\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n} & \mathbf{I}_{n_y}
\end{bmatrix}}_{3n\times 3n}
\underbrace{\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}}_{3n\times 3n}
\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix}$$

$$= \begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
-\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) & \mathbf{I}_n & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n} & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix} \mathbf{x}_t \\ -\mathbf{f}(\mathbf{r}_t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)-\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{I}_{n,n_y} & \mathbf{I}_{n,n_x} & \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{W}_t^{-1}\left(-\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t - \mathbf{f}(\mathbf{r}_t)\right) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

where $\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) \end{bmatrix}$ and we use the partioning $\mathbf{I}_n \equiv \begin{bmatrix} \mathbf{I}_{n,n_y} & \mathbf{I}_{n,n_x} \end{bmatrix}$ where $\mathbf{I}_{n,n_y}$ are the first n_y columns of $\mathbf{I}_n$ and $\mathbf{I}_{n,n_x}$ denotes the last columns of $\mathbf{I}_n$.

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

where $\begin{bmatrix} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{bmatrix} \equiv \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t)$ and $\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1}\left(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t)\right)$

$\Updownarrow$

$$\begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+1}) & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+1}) \\ \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\ \mathbf{0}_{n\times n_x} & -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+1}) & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+1}) & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{I}_{n_y} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ -\mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$\Updownarrow$

$$
\left[\begin{array}{cccccc}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{f_z}(\mathbf{r}_{t+1})-\mathbf{f_z}(\mathbf{r}_{t+1}) & \mathbf{f_x}(\mathbf{r}_{t+1})-\mathbf{f_x}(\mathbf{r}_{t+1}) & -\mathbf{f_z}(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t)-\mathbf{f_x}(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t)+\mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{array}\right.
$$

$$
=\left[\begin{array}{c}
\mathbf{x}_t \\
\mathbf{R}_{n_y}(t) \\
\mathbf{R}_{n_x}(t) \\
-\mathbf{f_z}(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t)-\mathbf{f_x}(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t)-\mathbf{f}(\mathbf{r}_{t+1}) \\
\mathbf{y}_{t+N}
\end{array}\right]
$$

$\Updownarrow$

$$
\left[\begin{array}{cccccc}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & -\mathbf{f_z}(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t)-\mathbf{f_x}(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t)+\mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2}
\end{array}\right]
$$

$$
=\left[\begin{array}{c}
\mathbf{x}_t \\
\mathbf{R}_{n_y}(t) \\
\mathbf{R}_{n_x}(t) \\
-\mathbf{f_z}(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t)-\mathbf{f_x}(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t)-\mathbf{f}(\mathbf{r}_{t+1}) \\
\mathbf{y}_{t+N}
\end{array}\right]
$$

Now let $\mathbf{W}_{t+1}\equiv\left[\begin{array}{cc} -\mathbf{f_z}(\mathbf{r}_{t+1})\mathbf{Q}_{n_y}(t)-\mathbf{f_x}(\mathbf{r}_{t+1})\mathbf{Q}_{n_x}(t)+\mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1})\end{array}\right]$ and $\left[\begin{array}{c}\mathbf{Q}_{n_y}(t+1)\\ \mathbf{Q}_{n_x}(t+1)\end{array}\right]\equiv\mathbf{W}_{t+1}^{-1}\mathbf{f_{yp}}(\mathbf{r}_{t+1})$ and $\left[\begin{array}{c}\mathbf{R}_{n_y}(t+1)\\ \mathbf{R}_{n_x}(t+1)\end{array}\right]\equiv-\mathbf{W}_{t+1}^{-1}\left(\mathbf{f_z}(\mathbf{r}_{t+1})\mathbf{R}_{n_y}(t)+\mathbf{f_x}(\mathbf{r}_{t+1})\mathbf{R}_{n_x}(t)+\mathbf{f}(\mathbf{r}_{t+1})\right)$. Using this we get

$$
\left[\begin{array}{cccccc}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2}
\end{array}\right]
=\left[\begin{array}{c}
\mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{y}_{t+N}
\end{array}\right]
$$

This system can then be solved by backward substitution as follows:

- $\mathbf{y}_{t+2}=\mathbf{y}_{t+N}$
- $\mathbf{x}_{t+2}=\mathbf{R}_{n_x}(t+1)-\mathbf{Q}_{n_x}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{y}_{t+1}=\mathbf{R}_{n_y}(t+1)-\mathbf{Q}_{n_y}(t+1)\mathbf{y}_{t+2}$
- $\mathbf{x}_{t+1}=\mathbf{R}_{n_x}(t)-\mathbf{Q}_{n_x}(t)\mathbf{y}_{t+1}$
- $\mathbf{y}_t=\mathbf{R}_{n_y}(t)-\mathbf{Q}_{n_y}(t)\mathbf{y}_{t+1}$

Recall that for period t

$$
\mathbf{W}_t\equiv\left[\begin{array}{cc}\mathbf{f_y}(\mathbf{r}_t) & \mathbf{f_{xp}}(\mathbf{r}_t)\end{array}\right]
$$

$$
\left[\begin{array}{c}\mathbf{Q}_{n_y}(t)\\ \mathbf{Q}_{n_x}(t)\end{array}\right]\equiv\mathbf{W}_t^{-1}\mathbf{f_{yp}}(\mathbf{r}_t)
$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{bmatrix} \equiv -\mathbf{W}_t^{-1}\left(\mathbf{f_x}(\mathbf{r}_t)\,\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t)\right)$$

and for period $t+1$

$$\mathbf{W}_{t+1} \equiv \begin{bmatrix} -\mathbf{f_z}(\mathbf{r}_{t+1})\,\mathbf{Q}_{n_y}(t) - \mathbf{f_x}(\mathbf{r}_{t+1})\,\mathbf{Q}_{n_x}(t) + \mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{Q}_{n_y}(t+1) \\ \mathbf{Q}_{n_x}(t+1) \end{bmatrix} \equiv \mathbf{W}_{t+1}^{-1}\mathbf{f_{yp}}(\mathbf{r}_{t+1})$$

$$\begin{bmatrix} \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \end{bmatrix} \equiv -\mathbf{W}_{t+1}^{-1}\left(\mathbf{f_z}(\mathbf{r}_{t+1})\,\mathbf{R}_{n_y}(t) + \mathbf{f_x}(\mathbf{r}_{t+1})\,\mathbf{R}_{n_x}(t) + \mathbf{f}(\mathbf{r}_{t+1})\right)$$

3.2 3 Time Periods

Let $N = 3$ and consider the system:

$$\underbrace{\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{f_x}(\mathbf{r}_t) & \mathbf{f_y}(\mathbf{r}_t) & \mathbf{f_{xp}}(\mathbf{r}_t) & \mathbf{f_{yp}}(\mathbf{r}_t) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{f_z}(\mathbf{r}_{t+1}) & \mathbf{f_x}(\mathbf{r}_{t+1}) & \mathbf{f_y}(\mathbf{r}_{t+1}) & \mathbf{f_{xp}}(\mathbf{r}_{t+1}) & \mathbf{f_{yp}}(\mathbf{r}_{t+1}) & \mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & \mathbf{f_z}(\mathbf{r}_{t+2}) & \mathbf{f_x}(\mathbf{r}_{t+2}) & \mathbf{f_y}(\mathbf{r}_{t+2}) & \mathbf{f_{xp}}(\mathbf{r}_{t+2}) & \mathbf{f_{yp}}(\mathbf{r}_{t+2}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}}_{4n\times 4n}
\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix}
= -\begin{bmatrix} -\mathbf{x}_t \\ \mathbf{f}(\mathbf{r}_t) \\ \mathbf{f}(\mathbf{r}_{t+1}) \\ \mathbf{f}(\mathbf{r}_{t+2}) \\ -\mathbf{y}_{t+N} \end{bmatrix}$$

Using the same transformations as above for the two first periods we have

$$\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{f_z}(\mathbf{r}_{t+2}) & \mathbf{f_x}(\mathbf{r}_{t+2}) & \mathbf{f_y}(\mathbf{r}_{t+2}) & \mathbf{f_{xp}}(\mathbf{r}_{t+2}) & \mathbf{f_{yp}}(\mathbf{r}_{t+2}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix}
= \begin{bmatrix} \mathbf{x}_t \\ -\mathbf{R}_{n_y}(t) \\ -\mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N} \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n\times n_x} & \mathbf{0}_{n\times n_y} & \mathbf{0}_{n\times n_x} & -\mathbf{f_z}(\mathbf{r}_{t+2}) & -\mathbf{f_x}(\mathbf{r}_{t+2}) & \mathbf{I}_n & \mathbf{0}_{n\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_y}
\end{bmatrix}$$

$$\times \begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_y} \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{f_z}(\mathbf{r}_{t+2}) & \mathbf{f_x}(\mathbf{r}_{t+2}) & \mathbf{f_y}(\mathbf{r}_{t+2}) & \mathbf{f_{xp}}(\mathbf{r}_{t+2}) & \mathbf{f_{yp}}(\mathbf{r}_{t+2}) \\
\mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+2} \\ \mathbf{y}_{t+2} \\ \mathbf{x}_{t+3} \\ \mathbf{y}_{t+3} \end{bmatrix}$$

$$
= \begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{0}_{n \times n_x} & -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2}) & -\mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & \mathbf{I}_{n} & \mathbf{0}_{n \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_y}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

$\Updownarrow$

$$
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2}) - \mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2}) & -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2})\mathbf{Q}_{n_y}(t+1) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\mathbf{Q}_{n_x}(t+1) + \mathbf{f}_{\mathbf{y}}(\mathbf{r} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2})\mathbf{R}_{n_y}(t+1) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\mathbf{R}_{n_x}(t+1) - \mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

$\Updownarrow$

$$
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{0}_{n \times n_x} & \mathbf{0}_{n \times n_y} & \mathbf{0}_{n \times n_x} & \mathbf{W}_{t+2} & - & \mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2}) \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2})\mathbf{R}_{n_y}(t+1) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\mathbf{R}_{n_x}(t+1) - \mathbf{f}(\mathbf{r}_{t+2}) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

Let $\mathbf{W}_{t+2} \equiv \begin{bmatrix} -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2})\mathbf{Q}_{n_y}(t+1) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\mathbf{Q}_{n_x}(t+1) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+2}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+2}) \end{bmatrix}$

$\Updownarrow$

$$
\begin{bmatrix}
\mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+1) & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+1) & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{Q}_{n_y}(t+2) \\
\mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_y} & \mathbf{I}_{n_x} & \mathbf{Q}_{n_x}(t+2) \\
\mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{0}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} & \mathbf{I}_{n_y}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{x}_t \\ \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \\ \mathbf{R}_{n_y}(t+1) \\ \mathbf{R}_{n_x}(t+1) \\ \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \\ \mathbf{y}_{t+N}
\end{bmatrix}
$$

where

$$\left[\begin{array}{c} \mathbf{Q}_{n_y}(t+2) \\ \mathbf{Q}_{n_x}(t+2) \end{array}\right] \equiv \mathbf{W}_{t+2}^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+2})$$

and

$$\left[\begin{array}{c} \mathbf{R}_{n_y}(t+2) \\ \mathbf{R}_{n_x}(t+2) \end{array}\right] \equiv -\mathbf{W}_{t+2}^{-1}\left(\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+2})\mathbf{R}_{n_y}(t+1) + \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+2})\mathbf{R}_{n_x}(t+1) + \mathbf{f}(\mathbf{r}_{t+2})\right).$$

Thus this system is easily solved by backwards substitution.

3.3 Summarizing: For N time periods

Thus the general solution is given by

- $\mathbf{y}_{t+N} = \mathbf{y}_{t+N}$ is given
- $\mathbf{y}_{t+k} = -\mathbf{R}_{n_y}(t+k) - \mathbf{Q}_{n_y}(t+k)\mathbf{y}_{t+k+1}$ for $k = 0, 1, 2, ..., N-1$
- $\mathbf{x}_{t+k+1} = -\mathbf{R}_{n_x}(t+k) - \mathbf{Q}_{n_x}(t+k)\mathbf{y}_{t+k+1}$ for $k = 0, 1, 2, ..., N-1$

The relevant matrices are given by:

$$\mathbf{W}_t \equiv \left[\begin{array}{cc} \mathbf{f}_{\mathbf{y}}(\mathbf{r}_t) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_t) \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{Q}_{n_y}(t) \\ \mathbf{Q}_{n_x}(t) \end{array}\right] \equiv \mathbf{W}_t^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_t)$$

$$\left[\begin{array}{c} \mathbf{R}_{n_y}(t) \\ \mathbf{R}_{n_x}(t) \end{array}\right] \equiv -\mathbf{W}_t^{-1}\left(\mathbf{f}_{\mathbf{x}}(\mathbf{r}_t)\mathbf{x}_t + \mathbf{f}(\mathbf{r}_t)\right)$$

and for period $t+k$

$$\mathbf{W}_{t+k} \equiv \left[\begin{array}{cc} -\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+k})\mathbf{Q}_{n_y}(t+k-1) - \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+k})\mathbf{Q}_{n_x}(t+k-1) + \mathbf{f}_{\mathbf{y}}(\mathbf{r}_{t+k}) & \mathbf{f}_{\mathbf{xp}}(\mathbf{r}_{t+k}) \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{Q}_{n_y}(t+k) \\ \mathbf{Q}_{n_x}(t+k) \end{array}\right] \equiv \mathbf{W}_{t+k}^{-1}\mathbf{f}_{\mathbf{yp}}(\mathbf{r}_{t+k})$$

$$\left[\begin{array}{c} \mathbf{R}_{n_y}(t+k) \\ \mathbf{R}_{n_x}(t+k) \end{array}\right] \equiv -\mathbf{W}_{t+k}^{-1}\left(\mathbf{f}_{\mathbf{z}}(\mathbf{r}_{t+k})\mathbf{R}_{n_y}(t+k-1) + \mathbf{f}_{\mathbf{x}}(\mathbf{r}_{t+k})\mathbf{R}_{n_x}(t+k-1) + \mathbf{f}(\mathbf{r}_{t+k})\right)$$

for $k = 1, 2, ..., N$

4 A New Keynesian Model

This section presents a standard New Keynesian model with Calvo pricing.

4.1 Households

The dynamic optimization problem faced by the representative household is of the form:

$$\underset{c_t, b_t, h_t, k_{t+1}, i_t \ \forall t \geq 0}{Max} \quad V_t = u\left(c_t, 1 - h_t\right)$$

$$\text{St.} \quad k_{t+1} = (1-\delta)\, k_t + i_t - k_t \tfrac{\kappa}{2} \left(\tfrac{i_t}{k_t} - \tfrac{i_{ss}}{k_{ss}} \right)^2$$

$$b_t + c_t + i_t = \tfrac{R_{t-1} b_{t-1}}{\pi_t} + h_t w_t + r_t^k k_t + div_t$$

a no-Ponzi-game condition

$c_t, h_t, k_{t+1}, i_t \geq 0 \ \forall t \geq 0$

As for the notation:

- c_t =consumption
- h_t =hours
- k_t = the capital stock
- i_t = investments
- R_t =the gross one-period risk-free rate, i.e. $R_t = 1 + r_t$
- b_t = nominal one-period bonds
- π_t = gross inflation of consumption good prices
- w_t = the wage level measured in consumption good units
- r_t^k = the rental rate for capital services sold to firms as measured in consumption good units
- div_t = net transfers of profit from firms to households as measured in consumption good units

The Lagranian is therefore given by:

$$\begin{aligned} \mathcal{L} = & \sum_{l=0}^{\infty} E_t \beta^l u\left(c_{t+l}, 1 - h_{t+l}\right) \\ & + E_t \sum_{l=0}^{\infty} \beta^l \lambda_{t+l} [\tfrac{R_{t-1+l}}{\pi_{t+l}} b_{t+l-1} + h_{t+l} w_{t+l} + r_{t+l}^k k_{t+l} + div_{t+l} - b_{t+l} - c_{t+l} - i_{t+l}] \\ & + E_t \sum_{l=0}^{\infty} \beta^l q_{t+l} \lambda_{t+l} \left[(1-\delta)\, k_{t+l} + i_{t+l} - k_{t+l} \tfrac{\kappa}{2} \left(\tfrac{i_{t+l}}{k_{t+l}} - \tfrac{i_{ss}}{k_{ss}} \right)^2 - k_{t+1+l} \right] \end{aligned}$$

That is, we introduce two lagrange multipliers:

- λ_t = for the budget constraint
- $q_t \lambda_t$ = for the capital accumulation equation

FOC (Kuhn-Tucker conditions)
We look for a solution in the interior, i.e. $c_t, b_t, h_t, k_{t+1}, i_t > 0 \ \forall t \geq 0$

1) Consumption, c_t :

$\frac{\partial \mathcal{L}}{\partial c_t} = u_c\left(c_t, 1-h_t\right) - \lambda_t = 0$
$\Updownarrow$
$\lambda_t = u_c\left(c_t, 1-h_t\right)$

2) Bonds, b_t :
$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_t + E_t\left[\beta\lambda_{t+1}\frac{R_t}{\pi_{t+1}}\right] = 0$
$\Updownarrow$

$1 = E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{R_t}{\pi_{t+1}}\right]$
Note that $D_{t,t+1} \equiv \beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}}$ is the nominal stochastic discount factor.

3) The labor supply, h_t
$\frac{\partial \mathcal{L}}{\partial h_t} = -u_{1-h}\left(c_t, 1-h_t\right) + \lambda_t w_t = 0$
$\Updownarrow$
$u_{1-h}\left(c_t, 1-h_t\right) = \lambda_t w_t$

3) The physical capital stock, k_{t+1} :
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t q_t + E_t\beta\lambda_{t+1}[r^k_{t+1} + q_{t+1}\left(1-\delta\right) \\ -q_{t+1}\frac{\kappa}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}\right)^2 + q_{t+1}\kappa\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}\right)\frac{i_{i+1}}{k^2_{t+1}}k_{t+1}] = 0$$
$\Updownarrow$
$$q_t\lambda_t = E_t\beta\lambda_{t+1}\left[r^k_{t+1} + q_{t+1}\left(1-\delta\right) - q_{t+1}\frac{\kappa}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}\right)^2 + q_{t+1}\kappa\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}\right)\frac{i_{i+1}}{k_{t+1}}\right]$$

4) Investments, i_t :
$\frac{\partial \mathcal{L}}{\partial i_t} = -\lambda_t + q_t\lambda_t\left(1 - \kappa\left(\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}}\right)\right) = 0$
$\Updownarrow$
$1 = q_t\left(1 - \kappa_2\left(\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}}\right)\right)$

4.2 Firms

4.2.1 Final Good Producers

The representative competitive consumption good producer chooses $y_{i,t}$ for $i \in [0,1]$ to solve:

$$\max_{y_{t,i}} \quad P_t y_t - \int_0^1 P_{i,t} y_{i,t} di$$
$$s.t.$$
$$y_t = \left(\int_0^1 \left(y_{i,t}\right)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}}.$$

As for the notation:

- $y_{i,t}$ =denotes the output from firm i at time t

- $P_{t,i}$ = the price of $y_{i,t}$.
- y_t = output from the final good producers
- P_t = price of y_t.

The first-order condition to the problem is given by:

$$y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t.$$

To find the expression for the aggregate price level we use the zero-profit condition, i.e.

$$\begin{aligned} P_t y_t &= \int_0^1 P_{i,t} y_{i,t} di \\ &= \int_0^1 P_{i,t} \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t di \\ &= y_t P_t^{\eta} \int_0^1 (P_{i,t})^{1-\eta} di \end{aligned}$$

$\Updownarrow$

$$P_t^{1-\eta} = \int_0^1 (P_{i,t})^{1-\eta} di$$

$\Updownarrow$

$$P_t = \left[\int_0^1 (P_{i,t})^{1-\eta} di\right]^{\frac{1}{1-\eta}}$$

4.2.2 Intermediate Good Producer

This section derives the first-order conditions for the ith firm's optimization problem. We start by deriving the equation for the real dividend payments:

$$div_{i,t} \equiv \left[\left(\frac{P_{i,t}}{P_t}\right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t}\right].$$

So the problem is

$$\underset{h_{i,t},k_{i,t},P_{i,t}\ \forall t\geq 0}{Max} \quad E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} div_{i,t+l}$$

St. $div_{i,t+l} \equiv \left(\frac{P_{i,t}}{P_t}\right)^{1-\eta} y_t - r_t^k k_{i,t} - w_t h_{i,t}$

$a_t k_{i,t}^{\theta} (z_t h_{i,t})^{1-\theta} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$

a no-Ponzi-game condition

Here, we use the following notation:

- r_t^k = the rental rate for capital services sold to firms as measured in consumption good units
- $k_{i,t}$ = the capital stock used by firm i at time t

- w_t = the wage level measured in consumption good units
- $h_{i,t}$ =hours used by firm i at time t
- a_t = stationary technology shocks

The law of motion for a_t is given by

$$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1},$$

where $\epsilon_{a,t+1} \sim \mathcal{NID}(0,1)$. The lagrangian is

$$\mathcal{L} = E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} [\left(\frac{P_{i,t+l}}{P_{t+l}}\right)^{1-\eta} y_{t+l} - r^k_{t+l} k_{i,t+l} - w_{t+l} h_{i,t+l}]$$
$$+ E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} mc_{i,t+l} \left[a_{t+l} k^{\theta}_{i,t+l} (h_{i,t+l})^{1-\theta} - \left(\frac{P_{i,t+l}}{P_{t+l}}\right)^{-\eta} y_{t+l}\right]$$

FOC (Kuhn-Tucker conditions)
We look for a solution in the interior, i.e. $h_{i,t}, k_{i,t}, P_{i,t} > 0 \; \forall \; t \geq 0$.

1) Demand for labor, $h_{i,t}$**:**
$\frac{\partial \mathcal{L}}{\partial h_{i,t}} = P_t \left(-w_t + mc_{i,t} a_t (1-\theta) k^{\theta}_{i,t} (h_{i,t})^{-\theta}\right)$
Since $P_t > 0$ we get
$mc_{i,t} a_t (1-\theta) k^{\theta}_{i,t} (h_{i,t})^{-\theta} = w_t$

2) Demand for capital, k$_{i,t}$**:**
$\frac{\partial \mathcal{L}}{\partial k_{i.t}} = P_t \left(-r^k_t + mc_{i,t} \theta a_t k^{\theta-1}_{i,t} (h_{i,t})^{1-\theta}\right)$

Since $P_t > 0$ we get
$r^k_t = mc_{i,t} \theta a_t k^{\theta-1}_{i,t} (h_{i,t})^{1-\theta}$

3) The optimal price, P$_{i,t}$**:**
We assume Calvo-pricing determined by α, giving the probability of a firm not being allowed to change its price in a given period. Notice, that all the re-optimizing firms face the same problem, hence they all set the same price. We denote this price by $\tilde{P}_t$. Hence, we have

$$\frac{P_{i,t+l}}{P_{t+l}} = \frac{\tilde{P}_t}{P_t \frac{P_{t+l}}{P_t}} = \frac{\tilde{P}_t \prod_{i=1}^{l} \frac{1}{\pi_{t+i}}}{P_t} \qquad \text{, where } \pi_{t+l} \equiv \frac{P_{t+l}}{P_{t+l-1}}.$$

If we only write the probability that the new price last forever, the Lagrangian reads

$$\mathcal{L} = E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l [\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)^{1-\eta} y_{t+l} - r^k_{t+l} k_{i,t+l} - w_{t+l} h_{i,t+l}]$$
$$+ E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l mc_{i,t+l} \left[a_{t+l} k^{\theta}_{i,t+l} (h_{i,t+l})^{1-\theta} - \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)^{-\eta} y_{t+l}\right]$$

The first-order-condition is:

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l [(1-\eta) \tilde{P}_t^{-\eta} P_t^{\eta-1} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)^{1-\eta} y_{t+l}$$

$+mc_{i,t+l}\eta\tilde{P}_t^{-\eta-1}P_t^{\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)^{-\eta}y_{t+l}]$

$$= E_t\sum_{l=0}^{\infty}D_{t,t+l}P_{t+l}\alpha^l\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)^{-\eta}y_{t+l}\left[(1-\eta)P_t^{-1}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)+\eta\frac{mc_{i,t+l}}{\tilde{P}_t}\right]$$

$$= E_t\sum_{l=0}^{\infty}D_{t,t+l}P_{t+l}\alpha^l\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)^{-\eta}y_{t+l}\left[\frac{(\eta-1)}{\eta}\frac{\tilde{P}_t}{P_t}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)-mc_{i,t+l}\right]\frac{-\eta}{\tilde{P}_t}$$

since $\eta > 1$ and $\tilde{P}_t > 0$, $\frac{\partial\mathcal{L}}{\partial\tilde{P}_t} = 0$ implies

$$E_t\sum_{l=0}^{\infty}D_{t,t+l}P_{t+l}\alpha^l\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)^{-\eta}y_{t+l}\left[\frac{(\eta-1)}{\eta}\frac{\tilde{P}_t}{P_t}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)-mc_{i,t+l}\right]=0$$

4.2.3 Marginal costs

We next show that marginal costs are identical across firms, i.e. $mc_{i,t} = mc_t$ for all i. Note from the first order conditions for $k_{i,t}$ and $h_{i,t}$ that

$\frac{mc_{i,t}a_tz_t(1-\theta)k_{i,t}^{\theta}(h_{i,t})^{-\theta}}{mc_{i,t}a_t\theta k_{i,t}^{\theta-1}(h_{i,t})^{1-\theta}} = \frac{w_t}{r_t^k}$

$\Updownarrow$

$\frac{(1-\theta)}{\theta k_{i,t}^{-1}(h_{i,t})} = \frac{w_t}{r_t^k}$

$\Updownarrow$

$\frac{1-\theta}{\theta}\frac{k_{i,t}}{h_{i,t}} = \frac{w_t}{r_t^k}$

implying that $\frac{k_{i,t}}{h_{i,t}}$ must be constant with respect to i. I.e.

$\frac{k_{i,t}}{h_{i,t}} = cons_t$

$\Updownarrow$

$k_{i,t} = h_{i,t}cons_t$

$\Downarrow$

$\int_0^1 k_{i,t}di = \int_0^1 h_{i,t}cons_t\ di$

$\Updownarrow$

$k_t = h_tcons_t$

$\Updownarrow$

$cons_t = k_t/h_t$

Hence,

$mc_{i,t}a_t\theta k_{i,t}^{\theta-1}(h_{i,t})^{1-\theta} = r_t^k$

$\Updownarrow$

$mc_{i,t}a_t\theta\left(\frac{k_{i,t}}{h_{i,t}}\right)^{\theta-1} = r_t^k$

$\Updownarrow$

$mc_{i,t}a_t\theta\left(\frac{k_t}{h_t}\right)^{\theta-1} = r_t^k$

This shows that $mc_{i,t} = mc_t$.

Hence, we can write the first order condition for $h_{i,t}$ and $k_{i,t}$ as

$$a_t(1-\theta)\left(\frac{h_t}{k_t}\right)^{-\theta}mc_t = w_t$$

$$r_t^k = \theta a_t \left(\frac{h_t}{k_t}\right)^{1-\theta} mc_t$$

4.2.4 The recursive representation of the price relation

We start by defining

$$x_t^1 \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l y_{t+l} mc_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$$x_t^2 \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l y_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

This implies that the first-order-condition for the price relation can be expressed as

$$E_t \sum_{l=0}^{\infty} D_{t,t+l}P_{t+l}\alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \left[\frac{\eta-1}{\eta}\frac{\tilde{P}_t}{P_t} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right) - mc_{t+l}\right] = 0$$

$\Updownarrow$

$$E_t \sum_{l=0}^{\infty} D_{t,t+l}P_{t+l}\alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \frac{(\eta-1)}{\eta}\frac{\tilde{P}_t}{P_t} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)$$
$$-E_t \sum_{l=0}^{\infty} D_{t,t+l}P_{t+l}\alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}}\right)^{-\eta} y_{t+l} mc_{t+l} = 0$$

$\Updownarrow$

$$E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l \tilde{P}_t \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta} y_{t+l}\frac{(\eta-1)}{\eta}$$
$$-E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l \tilde{P}_t \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta} y_{t+l} mc_{t+l} = 0 \quad \text{see below}$$

$\Updownarrow$

since $\tilde{P}_t > 0$

$$\underbrace{E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l y_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}}_{x_t^2} \frac{(\eta-1)}{\eta}$$

$$-\underbrace{E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l y_{t+l} mc_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}}_{x_t^1} = 0$$

$\Updownarrow$

$$x_t^2 \frac{(\eta-1)}{\eta} - x_t^1 = 0$$

$\Updownarrow$

$$\eta x_t^1 + (1-\eta)\, x_t^2 = 0$$

We used the fact that

$$P_{t+l}\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)^{1-\eta}=\tilde{P}_t^{1-\eta}P_{t+l}^{\eta}=\tilde{P}_t\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

and

$$P_{t+l}\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}}\right)^{-\eta}=\tilde{P}_t^{-\eta}P_{t+l}^{1+\eta}=\tilde{P}_t\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

The dynamic process for x_t^1

We define $\frac{\tilde{P}_t}{P_t}=\tilde{p}_t$. Therefore

$$\begin{aligned}
x_t^1 &= E_t\sum_{l=0}^{\infty}D_{t,t+l}\alpha^l y_{t+l}mc_{t+l}\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta} \\
&= y_t mc_{i,t}\tilde{p}_t^{-\eta-1}+E_t\sum_{l=1}^{\infty}D_{t,t+l}\alpha^l y_{t+l}mc_{t+l}\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta} \\
&= y_t mc_t\tilde{p}_t^{-\eta-1}+\tilde{p}_t^{-\eta-1}E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\tilde{p}_{t+1}^{\eta+1}\left(\frac{1}{\pi_{t+1}}\right)^{-\eta}x_{t+1}^1\right] \quad \text{, see below} \\
&= y_t mc_t\tilde{p}_t^{-\eta-1}+E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{1}{\pi_{t+1}}\right)^{-\eta}x_{t+1}^1\right]
\end{aligned}$$

At the third equatility sign we use the fact that

$$x_{t+1}^1 = E_{t+1}\sum_{l=0}^{\infty}D_{t+1,t+1+l}\alpha^l y_{t+1+l}mc_{t+1+l}\left(\frac{\tilde{P}_{t+1}}{P_{t+1}}\right)^{-\eta-1}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+1+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$change of index, $j=l+1$

$$x_{t+1}^1 = \tilde{p}_{t+1}^{-\eta-1}E_{t+1}\sum_{j=1}^{\infty}D_{t+1,t+j}\alpha^{j-1}y_{t+j}mc_{t+j}\prod_{i=1}^{j-1}\left(\frac{1}{\pi_{t+1+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$

$$x_{t+1}^1\tilde{p}_{t+1}^{1+\eta}\alpha D_{t,t+1}=E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}mc_{t+j}\prod_{i=2}^{j}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$

$$x_{t+1}^1\tilde{p}_{t+1}^{1+\eta}\alpha D_{t,t+1}\left(\frac{1}{\pi_{t+1}^{(1+\eta)/\eta}}\right)^{-\eta}=E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}mc_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$

$$x_{t+1}^1\tilde{p}_{t+1}^{1+\eta}\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}}\left(\frac{1}{\pi_{t+1}^{(1+\eta)/\eta}}\right)^{-\eta}=E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}mc_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

since $D_{t,t+1}=\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}}$

$\Updownarrow$

$$x_{t+1}^1\tilde{p}_{t+1}^{1+\eta}\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{1}{\pi_{t+1}}\right)^{-\eta}=E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}mc_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Downarrow$

$E_t\left[x^1_{t+1}\tilde{p}^{1+\eta}_{t+1}\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{1}{\pi_{t+1}}\right)^{-\eta}\right] = E_t\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}mc_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$
due to the law of iterated expectations, $E_t E_{t+1}[\cdot] = E_t[\cdot]$.

The dynamic process for x^2_t
$x^2_t = E_t\sum_{l=0}^{\infty}D_{t,t+l}\alpha^l y_{t+l}\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$

$= y_t\tilde{p}_t^{-\eta} + \tilde{p}_t^{-\eta}E_t\sum_{l=1}^{\infty}D_{t,t+l}\alpha^l y_{t+l}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$ since $\tilde{p}_t = \frac{\tilde{P}_t}{P_t}$

$= y_t\tilde{p}_t^{-\eta} + \tilde{p}_t^{-\eta}E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\tilde{p}^{\eta}_{t+1}\left(\frac{1}{\pi_{t+1}}\right)^{1-\eta}x^2_{t+1}\right]$, see below

$= y_t\tilde{p}_t^{-\eta} + E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{1}{\pi_{t+1}}\right)^{1-\eta}x^2_{t+1}\right]$

At the third equatility sign we use the fact that
$x^2_{t+1} = E_{t+1}\sum_{l=0}^{\infty}D_{t+1,t+1+l}\alpha^l y_{t+1+l}\left(\frac{\tilde{P}_{t+1}}{P_{t+1}}\right)^{-\eta}\prod_{i=1}^{l}\left(\frac{1}{\pi_{t+1+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
$\Updownarrow$change of index $j = 1 + l$

$x^2_{t+1} \quad = \tilde{p}_{t+1}^{-\eta}E_{t+1}\sum_{j=1}^{\infty}D_{t+1,t+j}\alpha^{j-1}y_{t+j}\prod_{i=1}^{j-1}\left(\frac{1}{\pi_{t+1+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
$\Updownarrow$

$x^2_{t+1}\alpha D_{t,t+1}\tilde{p}^{\eta}_{t+1} = E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}\prod_{i=2}^{j}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
$\Updownarrow$

$x^2_{t+1}\alpha D_{t,t+1}\tilde{p}^{\eta}_{t+1}\left(\frac{1}{\pi_{t+1}^{\eta/(\eta-1)}}\right)^{1-\eta} = E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
$\Updownarrow$

$x^2_{t+1}\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}}\tilde{p}^{\eta}_{t+1}\left(\frac{1}{\pi_{t+1}^{\eta/(\eta-1)}}\right)^{1-\eta} = E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
since $D_{t,t+1} = \beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}}$
$\Updownarrow$

$x^2_{t+1}\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\tilde{p}^{\eta}_{t+1}\left(\frac{1}{\pi_{t+1}}\right)^{1-\eta} = E_{t+1}\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$

$\Downarrow$

$E_t\left[x_{t+1}^2\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\tilde{p}_{t+1}^{\eta}\left(\frac{1}{\pi_{t+1}}\right)^{1-\eta}\right] = E_t\sum_{j=1}^{\infty}D_{t,t+j}\alpha^j y_{t+j}\prod_{i=1}^{j}\left(\frac{1}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
due to the law of iterated expectations, $E_tE_{t+1}[\cdot] = E_t[\cdot]$.

4.3 The Central Bank

We assume a standard Taylor rule of the form

$$\log\left(\frac{R_t}{R_{ss}}\right) = \rho_r\log\left(\frac{R_{t-1}}{R_{ss}}\right) + (1-\rho_r)\left(\kappa_\pi\log\left(\frac{\pi_t}{\pi_{ss}}\right) + \kappa_y\log\left(\frac{y_t}{y_{ss}}\right)\right)$$

4.4 Aggregation: Final good producer

From the final good producer we have
$y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$
$\Updownarrow$
$a_t k_{i,t}^{\theta} h_{i,t}^{1-\theta} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$

We next notice that:

1. $\int_0^1 h_{i,t}di \equiv h_t$,
2. $\int_0^1 k_{i,t}di \equiv k_t$
3. $cons_t = k_t/h_t$

Doing the summation with respect to i we get
$\int_0^1 a_t k_{i,t}^{\theta} h_{i,t}^{1-\theta} di = y_t \underbrace{\int_0^1\left(\frac{P_{i,t}}{P_t}\right)^{-\eta} di}_{s_{t+1}}$
$\Updownarrow$

$\int_0^1 a_t h_{i,t}\left(\frac{k_{i,t}}{h_{i,t}}\right)^{\theta} di = y_t s_{t+1}$
$\Updownarrow$

$\int_0^1 a_t h_{i,t}(\text{constant})^{\theta} di = y_t s_{t+1}$
$\Updownarrow$

$a_t(\text{constant})^{\theta}\int_0^1 h_{i,t}di = y_t s_{t+1}$
$\Updownarrow$

$a_t\left(\frac{k_t}{h_t}\right)^{\theta} h_t = y_t s_{t+1}$
$\Updownarrow$

$a_t (k_t)^\theta h_t^{1-\theta} = y_t s_{t+1}$
$\Updownarrow$

$a_t (k_t)^\theta (h_t)^{1-\theta} = y_t s_{t+1}$

and
$s_{t+1} \quad \equiv \int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} di$

$$= \underbrace{(1-\alpha)\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}}_{\text{opt. in period } t} + \underbrace{(1-\alpha)\,\alpha\left(\frac{\tilde{P}_{t-1}}{P_t}\right)^{-\eta}}_{\text{opt. in period } t-1} + \underbrace{(1-\alpha)\,\alpha^2\left(\frac{\tilde{P}_{t-2}}{P_t}\right)^{-\eta}}_{\text{opt. in period } t-2} + ...$$

$$= (1-\alpha)\sum_{j=0}^{\infty} \alpha^j \left[\frac{\tilde{P}_{t-j}}{P_t}\right]^{-\eta}$$

$$= (1-\alpha)\,\tilde{p}_t^{-\eta} + P_t^{\eta} \sum_{j=1}^{\infty} \alpha^j \left[\tilde{P}_{t-j}\right]^{-\eta} \qquad \text{where } \tilde{p}_{t-j} \equiv \frac{\tilde{P}_{t-j}}{P_t}$$

$$= (1-\alpha)\,\tilde{p}_t^{-\eta} + P_t^{\eta}\left(s_t \alpha P_{t-1}^{-\eta}\right) \qquad \text{,see below}$$

$$= (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\,(P_t/P_{t-1})^{\eta}\, s_t$$
$\Updownarrow$

$s_{t+1} = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\,(\pi_t)^{\eta}\, s_t$

We used that
$s_t = (1-\alpha)\sum_{j=0}^{\infty} \alpha^j \left[\frac{\tilde{P}_{t-j-1}}{P_{t-1}}\right]^{-\eta}$
$\Updownarrow$ change of index: $j+1=l$

$s_t P_{t-1}^{-\eta} = (1-\alpha)\sum_{l=1}^{\infty} \alpha^{l-1} \left[\tilde{P}_{t-l}\right]^{-\eta}$
$\Updownarrow$

$s_t P_{t-1}^{-\eta}\alpha = (1-\alpha)\sum_{l=1}^{\infty} \alpha^{l} \left[\tilde{P}_{t-l}\right]^{-\eta}$

So the ressource constraints in the goods market are:
1) $a_t k_t^\theta (h_t)^{1-\theta} = y_t s_{t+1}$

2) $s_{t+1} = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha \pi_t^{\eta} s_t$

4.5 The Aggregate Resource constraint

Recall that firm dividends in equilibrium are given by

$$\begin{aligned} div_{i,t} &\equiv \left[\left(\frac{P_{i,t}}{P_t}\right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t}\right] \\ &= y_t - r_t^k k_t - w_t h_t \end{aligned}$$

From the household budget constraint we have

$$b_t + c_t + i_t = \frac{R_{t-1} b_{t-1}}{\pi_t} + h_t w_t + r_t^k k_t + div_t$$

$\Updownarrow$

$$c_t + i_t = h_t w_t + r_t^k k_t + div_t$$

as bonds are in zero net supply in equilibrium. Inserting for firm dividends implies

$$\begin{aligned} c_t + i_t &= h_t w_t + r_t^k k_t + \left(y_t - r_t^k k_t - w_t h_t\right) \\ &= y_t \end{aligned}$$

4.6 The Goods Market: The Relation Between the Optimal Price and the Price Index

We start by noticing: the number of firms is by construct very large, so there is a fraction of $1-\alpha$ firms reoptimize their prices and the remaining fraction use the indexsation rule. This implies

$P_t \equiv \left[\int_0^1 P_{i,t}^{1-\eta} di\right]^{\frac{1}{1-\eta}}$

$\Updownarrow$

$P_t^{1-\eta} = \int_0^1 \left\{(1-\alpha)\tilde{P}_t^{1-\eta} + \alpha P_{i,t-1}^{1-\eta}\right\} di$

$= (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha \int_0^1 P_{i,t-1}^{1-\eta} di$

$= (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha \underbrace{\int_0^1 P_{i,t-1}^{1-\eta} di}_{P_{t-1}^{1-\eta}}$

$\Updownarrow$

$P_t^{1-\eta} = (1-\alpha)\tilde{P}_t^{1-\eta} + \alpha P_{t-1}^{1-\eta}$

$\Updownarrow$

$1 = (1-\alpha)\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta} + \alpha\left(\frac{P_{t-1}}{P_t}\right)^{1-\eta}$

$\Updownarrow$

$1 = (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}$, since $\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}$

4.7 The Functional Forms

The households' utility functions

$$u\left(c_t, 1-h_t\right) = \frac{1}{1-\phi_2}\left(c_t\right)^{1-\phi_2} + \phi_0 \frac{\left(1-h_t\right)^{1-\phi_1}}{1-\phi_1}$$

$$u_c\left(c_t, 1-h_t\right) = c_t^{-\phi_2}$$

$$u_{1-h}\left(c_t, 1-h_t\right) = \phi_0\left(1-h_t\right)^{-\phi_1}$$

4.8 Summarizing

	The Households
1	$\lambda_t = c_t^{-\phi_2}$
2	$q_t\lambda_t = E_t\beta\lambda_{t+1}[r_{t+1}^k + q_{t+1}\left(1-\delta\right) - q_{t+1}\frac{\kappa}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}\right)^2 + q_{t+1}\kappa\left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}}\right)\frac{i_{t+1}}{k_{t+1}}]$
3	$\phi_0\left(1-h_t\right)^{-\phi_1} = \lambda_t w_t$
4	$1 = q_t\left(1-\kappa_2\left(\frac{i_i}{k_t} - \frac{i_{ss}}{k_{ss}}\right)\right)$
5	$\lambda_t = \beta R_t E_t\left[\frac{\lambda_{t+1}}{\pi_{t+1}}\right]$
	The Firms
6	$mc_t a_t\left(1-\theta\right)\left(\frac{h_t}{k_t}\right)^{-\theta} = w_t$
7	$a_t mc_t\theta\left(\frac{h_t}{k_t}\right)^{1-\theta} = r_t^k$
8	$\frac{(\eta-1)x_t^2}{\eta} = y_t mc_t\tilde{p}_t^{-\eta-1} + E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{1}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)x_{t+1}^2}{\eta}\right]$
9	$x_t^2 = y_t\tilde{p}_t^{-\eta} + E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{1}{\pi_{t+1}}\right)^{1-\eta}x_{t+1}^2\right]$
10	$1 = \left(1-\alpha\right)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}$
	The Central Bank
11	$\log\left(\frac{R_t}{R_{ss}}\right) = \rho_r\log\left(\frac{R_{t-1}}{R_{ss}}\right) + \left(1-\rho_r\right)\left(\kappa_\pi\log\left(\frac{\pi_t}{\pi_{ss}}\right) + \kappa_y\log\left(\frac{y_t}{y_{ss}}\right)\right)$
	Other relations
12	$a_t k_t^\theta h_t^{1-\theta} = y_t s_{t+1}$
13	$s_{t+1} = \left(1-\alpha\right)\tilde{p}_t^{-\eta} + \alpha\pi_t^\eta s_t$
14	$k_{t+1} = \left(1-\delta\right)k_t + i_t - \frac{\kappa}{2}\left(\frac{i_i}{k_t} - \frac{i_{ss}}{k_{ss}}\right)^2 k_t$
15	$y_t = c_t + i_t$
	Exogenous processes
16	$\log a_{t+1} = \rho_a\log a_t + \sigma_a\epsilon_{a,t+1}$

4.9 The Intertemporal Elasticity of Substitution (IES)

The intertemporal elasticity of substitution (IES) is given by

$$IES = -\frac{u_c(t)}{c_t u_{cc}(t)}$$

$$= -\frac{\left(c_t\right)^{-\phi_2}}{c_t\left(-\phi_2 c_t^{\phi_2-1}\right)}$$

$$= \frac{1}{\phi_2}$$

4.10 The Frisch Labor Supply Elasticity

Recall that this elasticity is given by

$elas_F = \frac{u_h}{h\left(u_{hh} - \frac{u_{ch}}{u_{cc}}\right)}$

In our case

$u_h = -\phi_0 (1 - h_t)^{-\phi_1}$

$u_{hh} = -\phi_1 \phi_0 (1 - h_t)^{-\phi_1 - 1}$

$u_{ch} = 0$

So

$elas_F = \frac{u_h}{h\left(u_{hh} - \frac{u_{ch}}{u_{cc}}\right)} = \frac{-\phi_0 (1-h)^{-\phi_1}}{h\left(-\phi_1 \phi_0 (1-h)^{-\phi_1 - 1}\right)}$

$= \frac{-(1-h)^{-\phi_1}}{h\left(-\phi_1 (1-h)^{-\phi_1 - 1}\right)}$

$= \frac{1}{h\left(\phi_1 (1-h)^{-1}\right)}$

$= \frac{1}{\phi_1} \frac{1-h}{h}$

Hence, if $h = \frac{1}{3}$, then we get

$elas_F = \frac{1}{\phi_1} \frac{1-\frac{1}{3}}{\frac{1}{3}} = \frac{1}{\phi_1} \frac{3-1}{1} = \frac{2}{\phi_1}$.

Thus, we get a Frisch labor elasticity of one by letting $\phi_1 = 2$.

4.11 The Steady State

This section solves for the steady state as a function of the structural parameters. We denote variables in steady state by subscript ss. The steady state value of labor, i.e. h_{ss}, is assumed to be given and we then back out the value of ϕ_0. We also consider π_{ss} as given.

The optimal relative price, $\tilde{p}_{ss}$.

From EQ 10

$1 = (1-\alpha) \tilde{p}_t^{1-\eta} + \alpha \left(\frac{1}{\pi_t}\right)^{1-\eta}$

$\Downarrow$

$1 - \alpha \pi_{ss}^{-(1-\eta)} = (1-\alpha) \tilde{p}_{ss}^{1-\eta}$

$\Updownarrow$

$\tilde{p}_{ss} = \left[\frac{1 - \alpha \pi_{ss}^{(\eta-1)}}{1-\alpha}\right]^{\frac{1}{1-\eta}}$

The state variable for distortion due to price stickyness, s_{ss}

From EQ 13

$s_{t+1} = (1-\alpha) \tilde{p}_t^{-\eta} + \alpha \pi_t^{\eta} s_t$

$\Downarrow$

$s_{ss} = (1-\alpha) \tilde{p}_{ss}^{-\eta} + \alpha \pi_{ss}^{\eta} s_{ss}$

$\Updownarrow$

$s_{ss} (1 - \alpha \pi_{ss}^{\eta}) = (1-\alpha) \tilde{p}_{ss}^{-\eta}$

$\Updownarrow$

$s_{ss} = \frac{(1-\alpha) \tilde{p}_{ss}^{-\eta}}{1 - \alpha \pi_{ss}^{\eta}}$

The nominal one period interest rate, R_{ss}

From EQ 5

$1 = E_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\pi_{t+1}} \right]$

$\Downarrow$

$R_{ss} = \frac{\pi_{ss}}{\beta}$

The real price of capital , q_{ss}

From EQ 4

$1 = q_t \left(1 - \kappa \left(\frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right) \right)$

We immediately get that $q_{ss} = 1$

The real price of capital, r^k_{ss}

From EQ 2

$q_t \lambda_t = E_t \beta \lambda_{t+1} [r^k_{t+1} + q_{t+1} (1 - \delta) - q_{t+1} \frac{\kappa}{2} \left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + q_{t+1} \kappa \left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \frac{i_{t+1}}{k_{t+1}}]$

$\Downarrow$

$q_{ss} = \beta [r^k_{ss} + q_{ss} (1 - \delta)]$

$\Updownarrow$

$r^k_{ss} = q_{ss} \left(\frac{1}{\beta} - (1 - \delta) \right)$

The marginal costs in the firms, mc_{ss}

First from EQ 8

$\frac{(\eta-1) x^2_t}{\eta} = y_t mc_t \tilde{p}_t^{-\eta-1} + E_t \left[\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta-1} \left(\frac{1}{\pi_{t+1}} \right)^{-\eta} \frac{(\eta-1) x^2_{t+1}}{\eta} \right]$

$\Downarrow$

$\frac{(\eta-1) x^2_{ss}}{\eta} = y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1} + \alpha \beta \left(\frac{1}{\pi_{ss}} \right)^{-\eta} \frac{(\eta-1) x^2_{ss}}{\eta}$

$\Updownarrow$

$x^2_{ss} \frac{(\eta-1)}{\eta} \left(1 - \alpha \beta \left(\frac{1}{\pi_{ss}} \right)^{-\eta} \right) = y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1}$

$\Updownarrow$

$x^2_{ss} = \frac{y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1}}{\frac{(\eta-1)}{\eta} (1 - \alpha \beta \pi^{\eta}_{ss})}$

And from EQ 9

$x^2_t = y_t \tilde{p}_t^{-\eta} + E_t \left[\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta} \left(\frac{1}{\pi_{t+1}} \right)^{1-\eta} x^2_{t+1} \right]$

$\Downarrow$

$x^2_{ss} = y_{ss} \tilde{p}_{ss}^{-\eta} + \alpha \beta \left(\frac{1}{\pi_{ss}} \right)^{1-\eta} x^2_{ss}$

$\Updownarrow$

$x^2_{ss} = \frac{y_{ss} \tilde{p}_{ss}^{-\eta}}{1 - \alpha \beta \left(\frac{1}{\pi_{ss}} \right)^{1-\eta}}$

So by setting the two equations equal to one another we get:

$\frac{y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1}}{\frac{(\eta-1)}{\eta} (1 - \alpha \beta \pi^{\eta}_{ss})} = \frac{y_{ss} \tilde{p}_{ss}^{-\eta}}{1 - \alpha \beta \left(\frac{1}{\pi_{ss}} \right)^{1-\eta}}$

$\Updownarrow$

$mc_{ss} = \frac{\tilde{p}_{ss} \frac{(\eta-1)}{\eta} (1 - \alpha \beta \pi^{\eta}_{ss})}{1 - \alpha \beta \left(\frac{1}{\pi_{ss}} \right)^{1-\eta}}$

The Capital level, k_{ss}
From EQ 7
$a_t mc_t \theta \left(\frac{h_t}{k_t}\right)^{1-\theta} = r_t^k$
$\Downarrow$
$a_{ss} mc_{ss} \theta \left(\frac{h_{ss}}{k_{ss}}\right)^{1-\theta} = r_{ss}^k$
$\Updownarrow$
$\frac{1}{k_{ss}} = \left[\frac{r_{ss}^k}{a_{ss} mc_{ss} \theta}\right]^{\frac{1}{1-\theta}} \frac{1}{h_{ss}}$

$\Updownarrow$
$k_{ss} = \left[\frac{r_{ss}^k}{a_{ss} mc_{ss} \theta}\right]^{\frac{1}{\theta-1}} h_{ss}$

The wage level, W_{ss}
From EQ 7
$mc_t a_t (1-\theta) \left(\frac{h_t}{k_t}\right)^{-\theta} = w_t$
$\Downarrow$
$w_{ss} = mc_{ss} a_{ss} (1-\theta) \left(\frac{h_{ss}}{k_{ss}}\right)^{-\theta}$

The investment level, i_{ss}
From EQ 14
$k_{t+1} = (1-\delta) k_t + i_t - \frac{\kappa}{2} \left(\frac{i_i}{k_t} - \frac{i_{ss}}{k_{ss}}\right)^2 k_t$
$\Downarrow$
$k_{ss} = (1-\delta) k_{ss} + i_{ss}$
$\Updownarrow$
$i_{ss} = \delta k_{ss}$

The output level, y_{ss}
From EQ12
$a_t k_t^{\theta} h_t^{1-\theta} = y_t s_{t+1}$
$\Downarrow$
$a_{ss} k_{ss}^{\theta} h_{ss}^{1-\theta} = y_{ss} s_{ss}$
$\Updownarrow$
$y_{ss} = \frac{a_{ss} k_{ss}^{\theta} h_{ss}^{1-\theta}}{s_{ss}}$

The consumption level: c$_{ss}$
From EQ15
$y_t = c_t + i_t$
$\Downarrow$
$c_{ss} = y_{ss} - i_{ss}$

The value of λ_t
From EQ 1
$\lambda_{ss} = c_{ss}^{-\phi_2}$

The value of ϕ_0

From EQ 3

$\phi_0 (1 - h_t)^{-\phi_1} = \lambda_t w_t$

$\Downarrow$

$\phi_0 = \lambda_{ss} w_{ss} (1 - h_{ss})^{\phi_1}$

4.12 Model Equation for Accuracy Study

To evaluate accuracy, we use the following list of equations, where the residuals are expressed in unit-free terms ($\lambda_t = c_t^{-\phi_2}$)

	The Households
1	$1 = E_t \frac{\beta \lambda_{t+1}}{q_t \lambda_t} [r_{t+1}^k + q_{t+1} (1 - \delta) - q_{t+1} \frac{\kappa}{2} \left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + q_{t+1} \kappa \left(\frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \frac{i_{t+1}}{k_{t+1}}]$
2	$1 = \frac{\lambda_t w_t}{\phi_0 (1 - h_t)^{-\phi_1}}$
3	$1 = q_t \left(1 - \kappa_2 \left(\frac{i_i}{k_t} - \frac{i_{ss}}{k_{ss}} \right) \right)$
4	$1 = \beta R_t E_t \left[\frac{\lambda_{t+1}}{\lambda_t \pi_{t+1}} \right]$
	The Firms
5	$1 = \frac{mc_t a_t (1-\theta) \left(\frac{h_t}{k_t} \right)^{-\theta}}{w_t}$
6	$1 = \frac{a_t mc_t \theta \left(\frac{h_t}{k_t} \right)^{1-\theta}}{r_t^k}$
7	$1 = \frac{y_t mc_t \tilde{p}_t^{-\eta-1} + E_t \left[\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta-1} \left(\frac{1}{\pi_{t+1}} \right)^{-\eta} \frac{(\eta-1) x_{t+1}^2}{\eta} \right]}{\frac{(\eta-1) x_t^2}{\eta}}$
8	$1 = \frac{y_t \tilde{p}_t^{-\eta} + E_t \left[\alpha \beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta} \left(\frac{1}{\pi_{t+1}} \right)^{1-\eta} x_{t+1}^2 \right]}{x_t^2}$
9	$1 = (1 - \alpha) \tilde{p}_t^{1-\eta} + \alpha \left(\frac{1}{\pi_t} \right)^{1-\eta}$
	The Central Bank
10	$\log \left(\frac{R_t}{R_{ss}} \right) = \rho_r \log \left(\frac{R_{t-1}}{R_{ss}} \right) + (1 - \rho_r) \left(\kappa_\pi \log \left(\frac{\pi_t}{\pi_{ss}} \right) + \kappa_y \log \left(\frac{y_t}{y_{ss}} \right) \right)$
	Other relations
11	$1 = \frac{a_t k_t^\theta h_t^{1-\theta}}{y_t s_{t+1}}$
12	$1 = \frac{(1-\alpha) \tilde{p}_t^{-\eta} + \alpha \pi_t^\eta s_t}{s_{t+1}}$
13	$1 = \frac{(1-\delta) k_t + i_t - \frac{\kappa}{2} \left(\frac{i_i}{k_t} - \frac{i_{ss}}{k_{ss}} \right)^2 k_t}{k_{t+1}}$
14	$1 = \frac{c_t + i_t}{y_t}$
15	$1 = \frac{R_t}{(R_{t-1})_{t+1}}$, link equation for R_{t-1}
	Exogenous processes
16	$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$

4.13 The Upper Bound on Inflation

The Calvo model for staggered pricing implies an upper bound on inflation. To see this, recall that

$$P_t^{1-\eta} \equiv \int_0^1 P_{i,t}^{1-\eta} di = (1 - \alpha) \tilde{P}_t^{1-\eta} + \alpha P_{t-1}^{1-\eta}$$

$\Updownarrow$

$$1 = (1 - \alpha) \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\eta} + \alpha \left(\frac{1}{\pi_t} \right)^{1-\eta}.$$

Consider the limiting case, where marginal cost mc_t and hence the new price of the optimizing firm $\tilde{P}_t$ becomes arbitrary large. The constant elasticity of substitution index for prices with negative exponent, i.e. $P_t^{1-\eta} \equiv \int_0^1 P_{i,t}^{1-\eta} di$, implies that the aggregate price index remains bounded, because the components with large prices $(1-\alpha)\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta}$ tend to zero as $\eta > 1$. Isolating for inflation then gives the upper bound $\pi^{\max}$

$$1 = \alpha \left(\frac{1}{\pi^{\max}} \right)^{1-\eta}$$

$\Updownarrow$

$$\frac{1}{\pi^{\max}} = \left(\frac{1}{\alpha} \right)^{\frac{1}{1-\eta}}$$

$\Updownarrow$

$$\pi^{\max} = \left(\frac{1}{\alpha} \right)^{\frac{1}{\eta-1}}$$

Intuitively, consumers substitute away from the expensive goods with high prices to goods with old and lower prices. This substitution effect ensures that inflation across all goods in the economy remains bounded even when $\tilde{P}_t$ becomes arbitrary large.

Consider the case with price indexation of firms that do not re-optimize their prices. Then we have

$$P_t^{1-\eta} \equiv \int_0^1 P_{i,t}^{1-\eta} di = (1-\alpha)\, \tilde{P}_t^{1-\eta} + \alpha \left(\pi_{t-1}^{\chi} P_{t-1} \right)^{1-\eta}$$

$\Updownarrow$

$$1 = (1-\alpha) \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\eta} + \alpha \left(\frac{\pi_{t-1}^{\chi}}{\pi_t} \right)^{1-\eta}.$$

Thus, with full indexation ($\chi = 1$), we get an upper bound for the change in inflation as

$$1 = \alpha \left(\frac{\pi_{t-1}}{\pi_t} \right)^{1-\eta}.$$

$\Updownarrow$

$$\left(\frac{1}{\alpha} \right)^{1/(1-\eta)} = \frac{\pi_{t-1}}{\pi_t}$$

$\Updownarrow$

$$\log \left(\frac{\pi_t}{\pi_{t-1}} \right) = \log \left(\alpha^{1/(1-\eta)} \right).$$

5 A Multicountry RBC Model

We consider the multicountry RBC model presented in Juillard and Villemot (2011). The problem for the social planner is (with the same notation as in Juillard and Villemot (2011))

$$\underset{\left\{c_t^j, i^j, k_{t+1}^j\right\}_{t=0,1,\ldots,\infty.}^{j=1,2,\ldots,N}}{Max} \quad \mathbb{E}_t \sum_{j=1}^{N} \tau^j \left(\sum_{i=1}^{\infty} \beta^i \frac{\left(c_{t+i}^j\right)^{1-1/\gamma}}{1-1/\gamma} \right)$$

subject to

$$k_{t+1}^j = (1-\delta)\, k_t^j + i_t^j$$

$$\sum_{j=1}^N \left(c_t^j + i_t^j - \delta k_t^j\right) = \sum_{j=1}^N \left(a_t^j A \left(k_t^j\right)^\alpha - \frac{\phi}{2} k_t^j \left(\frac{i_t^j}{k_t^j} - \delta\right)^2\right)$$

The Lagranean reads

$$\begin{aligned}
\mathcal{L} \quad = \quad & \mathbb{E}_t \sum_{j=1}^N \tau^j \left(\sum_{i=1}^\infty \beta^i \frac{\left(c_{t+i}^j\right)^{1-1/\gamma}}{1-1/\gamma}\right) \\
& + \sum_{i=1}^\infty \beta^i \lambda_{t+i} q_{t+i} \left(k_{t+1+i}^j - (1-\delta)\, k_{t+i}^j - i_{t+i}^j\right) \\
& + \sum_{i=1}^\infty \beta^i \lambda_{t+i} \left(\sum_{j=1}^N \left(c_t^j + i_t^j - \delta k_t^j\right) - \sum_{j=1}^N \left(a_t^j A \left(k_t^j\right)^\alpha - \frac{\phi}{2} k_t^j \left(\frac{i_t^j}{k_t^j} - \delta\right)^2\right)\right)
\end{aligned}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t^j} : \tau^j \left(c_t^j\right)^{-1/\gamma} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial i_t^j} : -\lambda_t q_t + \lambda_t + \lambda_t \phi k_t^j \left(\frac{i_t^j}{k_t^j} - \delta\right) \frac{1}{k_t^j} = 0$$

$\Updownarrow$

$$\lambda_t \left[1 + \phi \left(\frac{i_t^j}{k_t^j} - \delta\right)\right] = \lambda_t q_t$$

$\Updownarrow$

$$q_t = 1 + \phi \left(\frac{i_t^j}{k_t^j} - \delta\right)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}^j} = \lambda_t q_t + \mathbb{E}_t \left[\beta \lambda_{t+1} \left(-q_{t+1} (1-\delta) - \delta - \alpha a_{t+1}^j A \left(k_{t+1}^j\right)^{\alpha-1} + \frac{\phi}{2} \left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta\right)^2 - \phi k_{t+1}^j \left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta\right) \frac{i_{t+1}^j}{\left(k_{t+1}^j\right)^2}\right)\right] = 0$$

$\Updownarrow$

$$\begin{aligned}
& \lambda_t \left[1 + \phi \left(\tfrac{i_t^j}{k_t^j} - \delta\right)\right] \\
& + \mathbb{E}_t \left[\beta \lambda_{t+1} \left(-\left[1 + \phi \left(\tfrac{i_{t+1}^j}{k_{t+1}^j} - \delta\right)\right] (1-\delta) - \delta - \alpha a_{t+1}^j A \left(k_{t+1}^j\right)^{\alpha-1} + \tfrac{\phi}{2} \left(\tfrac{i_{t+1}^j}{k_{t+1}^j} - \delta\right)^2 - \phi \left(\tfrac{i_{t+1}^j}{k_{t+1}^j} - \delta\right) \tfrac{i_{t+1}^j}{k_{t+1}^j}\right)\right] = 0
\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}
& \lambda_t \left[1 + \phi \left(\tfrac{i_t^j}{k_t^j} - \delta\right)\right] \\
& + \mathbb{E}_t \left[\beta \lambda_{t+1} \left(-(1-\delta) - \phi \left(\tfrac{i_{t+1}^j}{k_{t+1}^j} - \delta\right) (1-\delta) - \delta - \alpha a_{t+1}^j A \left(k_{t+1}^j\right)^{\alpha-1} + \tfrac{\phi}{2} \left(\tfrac{i_{t+1}^j}{k_{t+1}^j} - \delta\right)^2 - \phi \left(\tfrac{i_{t+1}^j}{k_{t+1}^j} - \delta\right) \tfrac{i_{t+1}^j}{k_{t+1}^j}\right)\right] = 0
\end{aligned}$$

$\Updownarrow$

$$\lambda_t \left[1 + \phi \left(\tfrac{i_t^j}{k_t^j} - \delta\right)\right]$$

$$+\mathbb{E}_t\left[\beta\lambda_{t+1}\left(-1-\alpha a^j_{t+1}A\left(k^j_{t+1}\right)^{\alpha-1}-\phi\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)(1-\delta)+\frac{\phi}{2}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)^2-\phi\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\frac{i^j_{t+1}}{k^j_{t+1}}\right)\right]=0$$

$\Updownarrow$

$$\lambda_t\left[1+\phi\left(\frac{i^j_t}{k^j_t}-\delta\right)\right]$$
$$+\mathbb{E}_t\left[\beta\lambda_{t+1}\left(-1-\alpha a^j_{t+1}A\left(k^j_{t+1}\right)^{\alpha-1}-\phi\left\{(1-\delta)+\frac{i^j_{t+1}}{k^j_{t+1}}-\frac{1}{2}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\right\}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\right)\right]=0$$

$\Updownarrow$

$$1+\phi\left(\frac{i^j_t}{k^j_t}-\delta\right)=\mathbb{E}_t\left[\frac{\beta\lambda_{t+1}}{\lambda_t}\left(1+\alpha a^j_{t+1}A\left(k^j_{t+1}\right)^{\alpha-1}+\phi\left\{(1-\delta)+\frac{i^j_{t+1}}{k^j_{t+1}}-\frac{1}{2}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\right\}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\right)\right]$$

Thus, the economy is given by

1	$\tau^j\left(c^j_t\right)^{-1/\gamma}=\lambda_t$
2	$\lambda_t\left[1+\phi\left(\frac{i^j_t}{k^j_t}-\delta\right)\right]=\mathbb{E}_t\left[\beta\lambda_{t+1}\left(1+\alpha a^j_{t+1}A\left(k^j_{t+1}\right)^{\alpha-1}+\phi\left\{(1-\delta)+\frac{i^j_{t+1}}{k^j_{t+1}}-\frac{1}{2}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\right\}\left(\frac{i^j_{t+1}}{k^j_{t+1}}-\delta\right)\right)\right]$
3	$k^j_{t+1}=(1-\delta)k^j_t+i^j_t$
4	$\sum_{j=1}^{N}\left(c^j_t+i^j_t-\delta k^j_t\right)=\sum_{j=1}^{N}\left(a^j_tA\left(k^j_t\right)^{\alpha}-\frac{\phi}{2}k^j_t\left(\frac{i^j_t}{k^j_t}-\delta\right)^2\right)$
5	$\log a^j_{t+1}=\rho\log a^j_t+\sigma e^j_t$

For the steady state, we have

- $a^j_{ss}=1$ from Eq 5
- $k^j_{ss}=1$ from an assumption in Juillard and Villemot (2011)
- $c^j_{ss}=a^j_{ss}A\left(k^j_{ss}\right)^{\alpha}=A$

From Eq 3
$k^j_{ss}=(1-\delta)k^j_{ss}+i^j_{ss}$
$\Updownarrow$
$1=1-\delta+i^j_{ss}$
$\Updownarrow$
$i^j_{ss}=\delta$

From Eq 2
$$1+\phi\left(\frac{i^j_{ss}}{k^j_{ss}}-\delta\right)=\frac{\beta\lambda_{t+ss}}{\lambda_{ss}}\left(1+\alpha a^j_{ss}A\left(k^j_{ss}\right)^{\alpha-1}+\phi\left\{(1-\delta)+\frac{i^j_{ss}}{k^j_{ss}}-\frac{1}{2}\left(\frac{i^j_{ss}}{k^j_{ss}}-\delta\right)\right\}\left(\frac{i^j_{ss}}{k^j_{ss}}-\delta\right)\right)$$
$\Updownarrow$
$1=\beta(1+\alpha A)$
$\Updownarrow$
$1=\beta+\alpha\beta A$
$\Updownarrow$
$A=\frac{1-\beta}{\alpha\beta}$

From Eq 4
$$\sum_{j=1}^{N}\left(c^j_{ss}+i^j_{ss}-\delta k^j_{ss}\right)=\sum_{j=1}^{N}\left(a^j_{ss}A\left(k^j_{ss}\right)^{\alpha}-\frac{\phi}{2}k^j_{ss}\left(\frac{i^j_{ss}}{k^j_{ss}}-\delta\right)^2\right)$$
$\Updownarrow$

$\sum_{j=1}^{N} c_{ss}^{j} = \sum_{j=1}^{N} A$
$\Updownarrow$
$\sum_{j=1}^{N} A = \sum_{j=1}^{N} A$
ok

From Eq 1
$\lambda_{ss} = \tau^{j} \left(c_{ss}^{j}\right)^{-1/\gamma}$

For τ^{j}, we follow Juillard and Villemot (2011) and let

$$\tau^{j} = \frac{1}{\left(c_{ss}^{j}\right)^{-1/\gamma}} = A^{1/\gamma}.$$

The considered calibration is given by $\gamma = 0.25$, $\delta = 0.025$, $\beta = 0.99$, $\alpha = 0.36$, $\rho = 0.99$, $\sigma = 0.01$, and $\phi = 0.5$.

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