Peso Problems in the Estimation of the C-CAPM

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Abstract

This paper shows that the consumption-based capital asset pricing model (C-CAPM) with low-probability disaster risk rationalizes pricing errors. We find that implausible estimates of risk aversion and time preference are not puzzling if market participants expect a future catastrophic change in fundamentals, which just happens not to occur in the sample (a 'peso problem'). A bias in structural parameter estimates emerges as a result of pricing errors in quiet times. While the bias essentially removes the pricing error in the simple models when risk-free rates are constant, time-variation may also generate large and persistent *estimated* pricing errors in simulated data. We also show analytically how the problem of biased estimates can be avoided in empirical research by resolving the misspecification in moment conditions.

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1 Introduction

It is a widespread perception that the workhorse of financial economics – the consumption-based capital asset pricing model (C-CAPM) of Rubinstein (1976), Lucas (1978), and Breeden (1979) – has fallen on hard times.[1] Most prominent is the failure to account for the equity premium for any plausible values of risk aversion, which has been referred to as the 'equity premium puzzle' (Mehra and Prescott, 1985). These limitations have given rise to a vast literature of promising C-CAPM extensions to achieve a better empirical performance. As a fly in the ointment, Lettau and Ludvigson (2009) show that leading extensions – including models with long-run risk and recursive preferences (Bansal and Yaron, 2004), habit formation (Campbell and Cochrane, 1999), and limiting participation (Guvenen, 2009) – cannot explain the large and persistent pricing errors found empirically (referred to as Euler equation (EE) errors). Even if this 'pricing error puzzle' has not received as much attention as the equity premium puzzle, an open question remains why the leading asset pricing models do fail on that particular dimension (Lettau and Ludvigson, 2009, p.255):

> "Unlike the equity premium puzzle, these large Euler equation errors cannot be resolved with high values of risk aversion. To explain why the standard model fails, we need to develop [...] models that can rationalize its large pricing errors."

Against this backdrop, our paper makes four key contributions. *First*, we show that a C-CAPM model with rare disasters in the spirit of Rietz (1988) and Barro (2006, 2009) – i.e. allowing for low-probability events causing infrequent but sharp contractions – not only explains the equity premium (as shown in Barro, 2006), but also rationalizes the large pricing errors found empirically. In fact, the puzzle is not about how to *rationalize* pricing errors, but rather how to generate *empirical* pricing errors. *Second*, we shed light on the source of pricing errors by providing analytical expressions for asset returns, the stochastic discount factor (SDF), and EE errors, both in an endowment economy and in a production economy with low-probability consumption/capital disasters. Our analytical results demonstrate that the EE errors are intimately linked to the poor empirical performance of the C-CAPM found in econometric studies. In particular, we elucidate *why* the parameter estimates for time preference and risk aversion tend to be severely biased in empirical studies. *Third*, we run extensive Monte Carlo simulations that demonstrate the impact of low-probability events on the plausibility of standard C-CAPM parameter estimates in small samples when the model is estimated, as is standard, by the generalized method of moments (GMM). We find that

[1] See Ludvigson (2011) for an excellent survey of the C-CAPM literature. Kroencke (2017) argues that the deficiencies of the classical model might be attributed to a failure to measure consumption correctly.

implausibly high estimates for the risk aversion and/or time preference parameters – exactly as found in the empirical literature – naturally arise if market participants expect a future catastrophic change in fundamentals, which just happen not to occur in the sample, or in other words, if the estimation is subject to a so-called peso problem.[2] *Fourth*, to address this issue, we suggest two simple corrections of the moment conditions, both implying more plausible empirical parameter estimates.

The novel result of this paper is to show that estimated EE errors may result – despite that GMM's objective is to minimize pricing errors – together with biased parameter estimates. We present an analytical investigation of EE errors in models with rare events and a data generating process which is able to generate estimated EE errors and biased parameter estimates of similar order of magnitudes as we observe in empirical data. Similar to the statistical approach for heavy-tailed distributions in Kocherlakota (1997), we show that by accounting for rare disasters in the C-CAPM, the model produces reasonable parameter estimates and pricing errors consistent with the empirical data.

We show that in small samples where consumption disasters just did not happen to occur, or more generally, where the sample frequency of disasters is not equal to their population frequency, the standard moment conditions are misspecified. This misspecification in turn typically leads to substantial biases of parameter estimates because the objective of GMM is to minimize the squared EE errors. In a simple endowment economy, this objective has two related but unpleasant properties: (i) it essentially removes the pricing errors through (ii) biased parameter estimates of risk aversion and time preference. Only in cases where the minimum of the GMM objective is not sufficiently close to zero, as is often the case in models with changing investment opportunities associated with time-varying interest rates, *estimated* pricing errors may occur.

In line with the results in Lettau and Ludvigson (2009), we illustrate that a model with long-run risk and recursive preferences is unable to generate estimated EE errors, while generating moderately biased estimates of the risk aversion coefficient. Overall, our results thus indicate that rare disaster risk is key for explaining the pricing error puzzle.

From a practical standpoint, we put forth several ways of how the biased estimates can be avoided in empirical research by resolving the misspecification in samples with peso problems. Our first proposal starts by assuming that the C-CAPM with rare events is the true data generating process. Then, we use the implied EE errors to remove any misspecification in the moment conditions. While elegant, this remedy is far from perfect in that is not model-free and depends on the assets under consideration. Our second remedy builds on Parker and

[2]The term 'peso problem' is interchangeably with the small-sample inference problems arising from these expected events. The phenomenon is named after events in the Mexican peso market (Lewis, 1992, p.142).

Julliard (2005) and includes a set of constants in the moment conditions that are intended to capture any disaster risk without the need of specifying a particular model.

Our work relates to the literature on the impact of peso problems on financial markets (cf. Veronesi, 2004).[3] While the role of unobserved regime shifts, fat-tailed shocks, and peso problems has been recognized already in earlier literature as a source of misspecification when a C-CAPM is fitted to the data (cf. Kocherlakota, 1997; Saikkonen and Ripatti, 2000), we go beyond the past literature in various ways. First of all, we cast the problem within the rare disaster framework of Rietz (1988) and Barro (2006). In their framework, asset prices reflect risk premia for infrequent and severe disasters in which consumption drops sharply. If disasters are expected by investors ex-ante (reflected in their decisions on consumption and portfolio choice), even if they happen not to occur in sample, a sizeable equity premium can materialize.[4] Using historical estimates of consumption disasters for a broad set of countries over a very long period, Barro (2006) shows that a calibrated version of the standard C-CAPM with rare events is able to explain the level of the US equity premium for plausible parameters of risk aversion. We add to this literature by showing that the rare disaster framework helps along two other dimensions: (i) explaining the observed pricing errors that empirical researchers typically encounter in finite sample when fitting a standard C-CAPM, and maybe even more importantly, (ii) explaining the implausible estimates of structural parameters often obtained in empirical work.

Our paper also relates to work on the estimation of consumption-based asset pricing models. When taking the C-CAPM to the data, the traditional approach is to estimate the model by GMM. While the advantage of this approach is that it does not rely on a specific structural model, it is sensitive to peso problems. For example, Saikkonen and Ripatti (2000) illustrate the poor performance of the GMM estimator in small and even relatively large samples in the presence of potential regime shifts. Several authors have sought to address the empirical weaknesses of the C-CAPM through alternative estimation approaches. In settings with rare disasters, information-theoretic and simulation-based approaches have been proposed

[3]It seems fair to say that the work of Barro (2006) has led to a resurgence of academic interest in the rare disasters hypothesis. Subsequent work includes Barro and Ursúa (2008); Barro (2009); Nakamura, Steinsson, Barro, and Ursúa (2013); Barro and Jin (2021). Recent work shows that rare event models help to explain many phenomena in macro-finance. Examples include: (i) time-series predictability and excess volatility in stock markets (e.g. Gabaix, 2008, 2012; Wachter, 2013), (ii) currency markets (e.g. Burnside, Eichenbaum, Kleshchelski, and Rebelo, 2011; Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan, 2009), (iii) options markets (e.g. Gabaix, 2008; Backus, Chernov, and Martin, 2011; Gabaix, 2012; Barro and Liao, 2019; Seo and Wachter, 2019), and (iv) business cycle dynamics (e.g. Gourio, 2012; Gourio, Siemer, and Verdelhan, 2013). Tsai and Wachter (2015) provide a comprehensive literature survey on rare events.

[4]This is related to the statement in Cochrane (2005, p.30) that the US economy and other countries with high historical equity premia may simply constitute very lucky cases of history. Brown, Goetzmann, and Ross (1995) consider the related issue of survivorship for inference in empirical finance.

(among others, Julliard and Ghosh, 2012; Nakamura, Steinsson, Barro, and Ursúa, 2013; Sönksen and Grammig, 2021), although at the expense of additional complexity. Drawing on the C-CAPM with rare events, we derive analytically the source of misspecification of the moment conditions that plagues the GMM estimation of the C-CAPM in finite samples. Therefore, we intentionally choose to work with the standard GMM approach, precisely to understand better why the approach fails and generates estimated EE errors and grossly biased parameters. This allows us to transparently show that the pricing error puzzle and the poor performance of the C-CAPM are related (an issue that has hitherto received little emphasis). It turns out that a more flexible version of the model which includes constants to capture the disaster risk helps in alleviating the problems in empirical work.

The remainder of the paper is organized as follows. Section 2 provides a definition of the EE errors and revisits the pricing puzzle. Section 3 presents the asset pricing models with rare events, provides analytical solutions for computing the EE errors in general equilibrium, and gives the intuition for our analytical, empirical, and simulation-based results. Section 4 shows the empirical performance of the C-CAPM with respect to EE errors and parameter estimates. We also show how to resolve the misspecification that implies more plausible values of risk aversion and time preference. It also contains Monte Carlo evidence showing that rare disaster models help in explaining several dimensions of the empirical weaknesses of the standard C-CAPM including large *empirical* pricing errors. Section 5 concludes.

2 Pricing errors

In this section we define Euler equation (EE) errors, and present a brief discussion of the empirical facts and puzzles encountered in the data.

2.1 Definitions

For analytical convenience, we consider the standard first-order condition implied by the canonical version of the C-CAPM with time-separable utility functions,

$$u'(C_t) = \beta E_t \left[u'(C_s) R_s \right], \quad u' > 0,\ u'' < 0, \quad s \geq t. \tag{1}$$

The optimality condition in (1) is referred to as the Euler equation. It determines the optimal path of consumption, C_t, given the gross return on savings (or any traded asset), R_s, and the time-discount factor, $\beta \equiv e^{-\rho(s-t)} \in (0,1)$, with ρ denoting the subjective rate of time preference. When pricing assets, we define the stochastic discount factor (SDF) as the process m_s/m_t such that for *any* security i with price P_t^i and payoff X_s^i at date $s \geq t$

$$m_t P_t^i = E_t \left[m_s X_s^i \right] \quad \Rightarrow \quad 1 = E_t \left[(m_s/m_t) R_s^i \right], \tag{2}$$

in which $R^i_s \equiv X^i_s/P^i_t$ denotes the security's return. Hence, by comparing condition (1) to the pricing equation (2), we conclude that in the C-CAPM we discount the expected payoff of any given asset by $m_s/m_t = \beta u'(C_s)/u'(C_t)$ in order to find its equilibrium price.

Following Lettau and Ludvigson (2009), any deviations from (2), i.e.,

$$e^i_R \equiv E_t\left[(m_s/m_t)R^i_s\right] - 1, \quad e^i_{R,s} \equiv (m_s/m_t)R^i_s - 1, \tag{3}$$

$$e^i_X \equiv E_t\left[(m_s/m_t)(R^i_s - R^b_s)\right], \quad e^i_{X,s} \equiv (m_s/m_t)(R^i_s - R^b_s), \tag{4}$$

define Euler equation (EE) errors, based on the gross return on any tradable asset, R^i_s, or as a function of excess returns over a reference asset, $R^i_s - R^b_s$, for example the return on bonds. In what follows, we refer to either e^i_R or e^i_X as pricing error, whereas to their empirical counterparts $\widehat{e^i_R}$ and $\widehat{e^i_X}$ as the *estimated* pricing error. The latter is defined for specific preferences. For example, for the C-CAPM with power utility and risk aversion $\gamma > 0$, the EE errors for $s = t + 1$ are given by

$$\widehat{e^i_R} \equiv E_t\big[\hat{\beta}(C_{t+1}/C_t)^{-\hat{\gamma}}R^i_{t+1}\big] - 1, \quad \widehat{e^i_{R,t+1}} \equiv \hat{\beta}(C_{t+1}/C_t)^{-\hat{\gamma}}R^i_{t+1} - 1, \tag{5}$$

$$\widehat{e^i_X} \equiv E_t\big[\hat{\beta}(C_{t+1}/C_t)^{-\hat{\gamma}}(R^i_{t+1} - R^b_{t+1})\big], \quad \widehat{e^i_{X,t+1}} \equiv \hat{\beta}(C_{t+1}/C_t)^{-\hat{\gamma}}(R^i_{t+1} - R^b_{t+1}). \tag{6}$$

where $\hat{\beta}$ and $\hat{\gamma}$ denote the *estimated* parameters of time-preference and risk aversion. Using data on consumption and asset returns, these estimates are traditionally obtained by the generalized method of moments of Hansen (1982), minimizing a quadratic objective of the pricing errors. The fit of the model is often assessed by the root mean squared error (RMSE), which is a summary measure of the magnitude of the estimated EE errors.

2.2 Empirical puzzles

It is well established that the C-CAPM with power utility fails in several dimensions, in particular is incapable of explaining cross-sectional variation in average asset returns. Using US postwar data, Lettau and Ludvigson (2009, Table 1) show that the model generates substantial pricing errors. Moreover, the parameter estimates of the standard C-CAPM pricing kernel $m_{t+1}/m_t = \beta(C_{t+1}/C_t)^{-\gamma}$ are flawed: $\hat{\beta} = 1.41$, $\hat{\gamma} = 89.8$ in the case of two asset returns, and $\hat{\beta} = 1.39$, $\hat{\gamma} = 87.2$ when adding 6 size and book-to-market portfolios.

These estimates are inconsistent with standard parameterizations in macro and finance. A time-discount factor above one implies that households value future consumption *more* than current consumption, and the estimated parameter of relative risk aversion is far *higher* than the microeconomic evidence on individuals' behavior in risky gambles.

In addition, the pricing errors are economically large. The estimated EE error amounts to 2.71% (3.05%) p.a. for the two-assets case (the larger cross-section), leaving a substantial fraction of the cross-sectional variation of average returns unexplained. It is puzzling to the econometrician why individuals seem to accept surprisingly large and persistent EE errors. Economically, this result implies that consumers seem to accept a 2.5 dollar pricing error for each 100 dollar spent. Further, it is not possible to reduce the empirical EE error to smaller magnitudes (or even to zero) by choosing other parameter constellations. Additionally, Lettau and Ludvigson (2009) convincingly argue that many of the newly proposed extensions to the C-CAPM, in particular the prominent long-run risk model (Bansal and Yaron, 2004), are not capable of rationalizing the large pricing errors of the canonical model.

3 Asset pricing with rare events

In this section we present two asset pricing models to compute equilibrium consumption growth and asset returns and that provide our framework to rationalize EE errors: the first is a simple Lucas' endowment model with constant investment opportunities; the second is an endowment economy with changing investment opportunities that mimics the equilibrium dynamics of a production economy. Both models shed light on different aspects of the puzzle and provide directions for empirical research. A complete description of the models and of their equilibrium implications can be found in Appendices A.2 and A.3.

3.1 Lucas' endowment economy with rare events

Consider a representative-agent, fruit-tree economy of asset pricing with exogenous and stochastic production. Similar to Barro (2006), production is subject to rare events in the form of large negative shocks arriving at a constant rate. Gabaix (2008) and Wachter (2013) consider the case of time-varying arrival rates and recursive preferences.[5]

Technology. Suppose that production of perishable output, Y_t, is exogenously given (cf. Lucas, 1978): no resources are utilized, and there is no possibility of affecting the output at any time. The law motion of Y_t follows the Markov process

$$dY_t = \mu_t Y_t dt + \sigma_t Y_t dB_t + (Y_t - Y_{t-}) dN_t, \tag{7}$$

where μ_t denotes the drift, σ_t represents the size of the shocks, B_t is a standard Brownian motion, and N_t is a Poisson process with arrival rate λ. In the simple endowment economy, we set $\mu_t \equiv \bar{\mu}$ and $\sigma_t \equiv \bar{\sigma}$. Moreover, the jump size is set to be proportional to the value of output an instant before the jump, $Y_t - Y_{t-} \equiv (\exp(\bar{\nu}) - 1) Y_{t-}$, $\bar{\nu} < 0$, where Y_{t-} denotes

[5]Favero, Ortu, Tamoni, and Yang (2020) show that such rare disaster models satisfy the Hansen and Jagannathan (1991) bounds. However, both features are not required in generating EE errors.

the left-limit, $Y_{t-} \equiv \lim_{\tau \to t} Y_\tau$, for $\tau < t$, ensuring that Y_t does not jump negative. Thus, the investment opportunity set in this endowment economy is assumed to be constant.

Preferences. Consider an economy with a single consumer, interpreted as a representative of a large number of identical consumers. The representative consumer maximizes expected lifetime utility (cf. Svensson, 1989; Duffie and Epstein, 1992b),

$$U_0 \equiv E_0 \int_0^\infty f(C_t, U_t) dt \tag{8}$$

where $f(C_t, U_t)$ is a normalized aggregator given by

$$f(C_t, U_t) = \frac{1-\gamma}{1-1/\psi} U_t \left(C_t^{1-1/\psi} - \rho((1-\gamma)U_t)^{\frac{1-1/\psi}{1-\gamma}} \right) ((1-\gamma)U_t)^{-\frac{1-1/\psi}{1-\gamma}} \tag{9}$$

with $\rho > 0$ the subjective rate of time preference, $\gamma > 0$ the coefficient of relative risk aversion, and $\psi > 0$ the elasticity of intertemporal substitution.[6]

The general utility function (8) is introduced here to compare our rare events models with the main competitor, the long-run risk model of Bansal and Yaron (2004). To obtain analytical results in the rare event models we set $1/\psi = \gamma$, but our insights for the pricing errors do not depend on the parametric restriction, which can be relaxed at the cost of analytical tractability. Setting $1/\psi = \gamma$ makes the recursion in (9) linear, and the preferences in (8) collapse to the standard time-additive model with

$$f(C_t, U_t) = \frac{C_t^{1-\gamma}}{1-\gamma} - \rho U_t \quad \Rightarrow \quad U_0 \equiv E_0 \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt. \tag{10}$$

Equilibrium. In this economy, it is straightforward to determine equilibrium quantities and the equilibrium asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$, and all shares held by capital owners are in zero net supply.

3.2 A mimicking endowment economy with rare events

Consider an endowment economy mimicking the equilibrium dynamics of a production economy that is subject to rare disasters in the accumulation of the capital stock as in Gourio (2012), and rare technological improvements (cf. Wälde, 2005).[7] The production economy underlying this mimicking endowment economy is described in Appendix A.3. Below we use

[6]In contrast to Duffie and Epstein (1992a) and Wachter (2013), here ρ multiplies with the second part. Both formulations generate ordinal-equivalent utility functions, and hence the same equilibrium dynamics, though the value function may differ by a scale factor (cf. Appendix A.2.2).

[7]Embedding disasters in a general equilibrium production economy with heterogeneous firms induces strong nonlinearity, which helps replicating the failure of the C-CAPM in explaining the value premium in samples in which disasters are not materialized (cf. Bai, Hou, Kung, Li, and Zhang, 2019).

the terms mimicking endowment economy and production economy interchangeably, but the equilibrium dynamics coincide only for a particular ρ (as shown in Proposition A.9).

Technology. Suppose that production of perishable output, Y_t, is exogenously given: there is no possibility of affecting the output at any time. Let $Y_t = (1-s)A_t K_t^\alpha$, where K_t is the aggregate capital stock, A_t is the stochastic technology, $s \in (0,1)$ is the constant propensity to consume, and $\alpha \in (0,1)$ is the capital share in aggregate output.[8]

The law of motion of A_t will be taken to follow a Markov process, driven by a standard Brownian motion B_t, and a Poisson process $\bar{N}_t$ with arrival rate $\bar{\lambda}$,

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t d\bar{B}_t + (\exp(\bar{\nu}) - 1) A_{t-} d\bar{N}_t. \tag{11}$$

We introduce jumps in technology as there is empirical evidence of Poisson jumps in output growth. However, these may not necessarily reflect consumption disasters (Posch, 2009).

The aggregate capital stock is subject to stochastic depreciation,

$$dK_t = (s A_t K_t^\alpha - \delta K_t)dt + \sigma K_t dZ_t + (\exp(\nu) - 1) K_{t-} dN_t, \tag{12}$$

where $\delta > 0$ is the depreciation rate of physical capital, Z_t is a standard Brownian motion, and N_t is a Poisson process (both uncorrelated with $\bar{B}_t$ and $\bar{N}_t$) with constant arrival rate λ and $\nu < 0$. The jump size in capital is assumed to be proportional to the value of the capital stock an instant before a disaster, and it has a degenerated distribution. For $\sigma = \nu = 0$, the capital stock (physical asset) would be instantaneously riskless (cf. Merton, 1975).

Hence, output in this mimicking endowment economy follows

$$dY_t \;=\; \mu_t Y_t dt + \sigma_t Y_t dB_t + (Y_t - Y_{t-})(d\bar{N}_t + dN_t) \tag{13}$$

with $\mu_t = \bar{\mu} + s r_t - \alpha\delta + \frac{1}{2}\alpha(\alpha-1)\sigma^2$, $\sigma_t \equiv \bar{\sigma}$, $B_t = \bar{B}_t + \sigma\alpha/\bar{\sigma} Z_t$, and where $r_t = \alpha A_t K_t^{\alpha-1}$ evolves according to

$$\begin{aligned} dr_t \;&=\; c_1\big(c_2 - r_t\big) r_t dt + (\alpha-1)\sigma r_t dZ_t + \bar{\sigma} r_t d\bar{B}_t \\ &\quad +(\exp((\alpha-1)\nu) - 1) r_{t-} dN_t + (\exp(\bar{\nu}) - 1) r_{t-} d\bar{N}_t \end{aligned} \tag{14}$$

with $c_1 \equiv \frac{1-\alpha}{\alpha\gamma}$ and $c_2 \equiv \alpha\gamma\delta - \frac{1}{2}\alpha\gamma(\alpha-2)\sigma^2 - \frac{\alpha\gamma}{\alpha-1}\bar{\mu}$. Note that the mimicking endowment economy introduces a stochastically changing investment opportunity set through μ_t.

Preferences. The representative consumer maximizes expected discounted lifetime utility given in (8) and (9). Further assume that $1/\psi = \gamma$ such that the problem is reduced to the standard power utility case in (10).

[8] For convenience, we set $Y_t = C_t$ in the mimicking endowment economy, though 'output' in the production economy is higher by a factor $1/(1-s)$. In the Online Appendix we present an alternative mimicking economy where the endowment is a linear function of the aggregate capital stock, $Y_t = kK_t$ with $k > 0$.

Equilibrium. In this economy, it is straightforward to determine equilibrium quantities and the equilibrium asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$, and households own the physical capital. All other assets are zero in net supply.

4 Results

In what follows, the equilibrium consumption growth rates and asset returns are used to compute EE errors in both the endowment and production economy. Moreover, we provide directions for empirical research for using GMM, and report empirical estimates.

Let ϕ denote the parameter vector and $h_t = h_t(\phi)$ a vector of martingale increments (moment conditions) in terms of data and parameters generated by the model. Further, let

$$H_T(\phi) = \sum_{t=1}^{T} h_t(\phi),$$

so that $H_T(\phi)/T$ is the sample average of $\{h_t(\phi)\}$. We define the GMM estimator as the minimizer of the squared norm $H_T(\phi)^\top W H_T(\phi)$, where W is a weight matrix, using the identity matrix $I_{\dim(h)}$ for W in a first-step minimization, and the estimates, say $\hat{\phi}_0$, to calculate an estimate of $Var(H_T(\hat{\phi}_0))^{-1}$, which then defines $W = (\sum_t h_t(\hat{\phi}_0) h_t(\hat{\phi}_0)^\top)^{-1}$ in the next-step minimization (cf. Hansen, 1982; Christensen, Posch, and van der Wel, 2016).

Without loss of generality, we consider annual returns, such that $s = t + 1$. For the case of two assets, e.g., a bond with return R^b_{t+1}, and an asset with return R^c_{t+1}. Then, for a given sample of data $h_{t+1}(\phi)$ can be constructed for the canonical C-CAPM model using (3):

$$h_{t+1}(\phi) = \begin{pmatrix} e^b_{R,t+1} \\ e^c_{R,t+1} \end{pmatrix} = \begin{pmatrix} \beta(C_{t+1}/C_t)^{-\gamma} R^b_{t+1} - 1 \\ \beta(C_{t+1}/C_t)^{-\gamma} R^c_{t+1} - 1 \end{pmatrix}, \tag{15}$$

or, alternatively using (4):

$$h_{t+1}(\phi) = \begin{pmatrix} e^c_{X,t+1} \\ e^i_{R,t+1} \end{pmatrix} = \begin{pmatrix} \beta(C_{t+1}/C_t)^{-\gamma} (R^c_{t+1} - R^b_{t+1}) \\ \beta(C_{t+1}/C_t)^{-\gamma} R^i_{t+1} - 1 \end{pmatrix},$$

based on excess returns and one additional asset i, where $\phi = (\beta, \gamma)^\top$.

Although previous work has emphasized that GMM is not the optimal choice for the econometric estimation in the presence of unobserved regime shifts and peso problems (Kocherlakota, 1997; Saikkonen and Ripatti, 2000), we estimate the unknown parameters $\phi = (\beta, \gamma)^\top$ of the canonical C-CAPM using the standard GMM approach. The main reason is that we are not primarily interested in estimating the rare disaster model. Instead, we ask whether we can replicate the empirical patterns by using the GMM approach (cf. Lettau and Ludvigson, 2009) and the potentially misspecified moment conditions (15).

4.1 Euler equation errors in finite samples

In this section, we show that pricing errors can be substantial in finite samples. We also illustrate how pricing errors may emerge as a result of various types of rare events, such as default risk, and/or rare improvements in the technology frontier.

Proposition 4.1 (Endowment economy) *Consider the model outlined in Section 3.1 with preferences in* (10) *such that the utility function exhibits constant relative risk aversion* γ*. Then, the Euler equation error (3), when conditioning on samples without disasters, on*

1. *the asset with payoff* $X^c_{t+1} = Y_{t+1}$ *and return* $R^c_{t+1} = X^c_{t+1}/P^c_t$ *is*

$$e^c_{R|N_{t+1}-N_t=0} = \exp\left((1 - e^{(1-\gamma)\bar{\nu}})\lambda\right) - 1; \tag{16}$$

2. *the bond with payoff* $X^b_{t+1} = e^{\int_t^{t+1} \ln(1+D_s)dN_s}$ *subject to default risk* $D_s = e^{\kappa} - 1$ *where* $\kappa < 0$ *with probability* q *in case of a disaster and return* $R^b_{t+1} = 1/P^b_t$ *is*

$$e^b_{R|N_{t+1}-N_t=0} = \exp\left((1 - e^{-\bar{\nu}\gamma})\lambda + e^{-\gamma\bar{\nu}}(1 - e^{\kappa})q\lambda\right) - 1, \tag{17}$$

where κ *is the size of the default, and* $q \in [0, 1]$ *is the probability of default;*

3. *the risk-free asset with payoff* $X^f_{t+1} = 1$ *and return* $R^f_{t+1} = 1/P^f_t$ *is*

$$e^f_{R|N_{t+1}-N_t=0} = \exp\left((1 - e^{-\bar{\nu}\gamma})\lambda\right) - 1. \tag{18}$$

Proof. See Appendix A.2.4. ■

For $\lambda = 0.017$ and $\bar{\nu} = -0.4$ (cf. Barro, 2006), the absolute EE error on a risky claim is about 3.9% for $\gamma = 4$ and would further increases with risk aversion. We argue that the EE errors can be large in finite samples and that empirical pricing errors measure disaster risk. Our results in Proposition 4.1 show that individuals would accept persistent pricing errors for the events that happen not to occur in normal times.[9] In fact, the probability of no disaster occurring in a randomly selected sample of 50 years is $p(N_{t+T} - N_t = 0) = e^{-\lambda T} = 43\%$.

Based on the excess return of a risky asset over a bond, we obtain (4) as

$$e^c_{X|N_{t+1}-N_t=0} = e^c_{R|N_{t+1}-N_t=0} - e^b_{R|N_{t+1}-N_t=0} = e^{(1-e^{(1-\gamma)\bar{\nu}})\lambda} - e^{(1-e^{-\bar{\nu}\gamma})\lambda + e^{-\gamma\bar{\nu}}(1-e^{\kappa})q\lambda}.$$

Hence, any default risk would also rationalize the occurrence of EE error in quiet times. However, the default risk per se only generates pricing errors for the bond.[10] These results

[9] This result relates to Hansen and Jagannathan (1991, p.250), who note that the sample volatility may be substantially different than the population volatility if consumers anticipate that extremely bad events can occur with small probability, even when such events do not occur in the sample.

[10] For the extreme case $\kappa \to -\infty$ (complete default) and default with probability $q = 1$ in case of a disaster, the rational pricing error is approximately λ, i.e., the arrival rate of disasters/defaults.

point to an interesting direction for empirical research because, by having more assets in the sample, we may identify additional parameters including the default risk q. Below we estimate the disaster risk along with the standard parameters (cf. Table A.2).

One alternative interpretation of the pricing errors is through the implicit risk premium, i.e., the difference between the expected value of an uncertain rate of return and the certainty equivalent rate of return which makes an individual indifferent between both assets,

$$RP = \gamma\bar{\sigma}^2 + e^{-\gamma\bar{\nu}}(1 - e^{\bar{\nu}})\lambda = \gamma\bar{\sigma}^2 + \log\big((1 + e^c_R)/(1 + e^f_R)\big). \tag{19}$$

Hence, in the endowment economy, the ratio of pricing errors in (19) measures the implicit risk premium RP related to the disaster risk (cf. Posch, 2011).

Proposition 4.2 (Production economy) *Consider the model outlined in Section 3.2 with preferences in* (10) *such that the utility function exhibits constant relative risk aversion* γ*. Then, the Euler equation error (3), when conditioning on samples without disasters, on*

1. *the asset with payoff* $X^c_t = K^{\alpha\gamma}_{t+1}$ *and return* $R^c_{t+1} = K^{\alpha\gamma}_{t+1}/P^c_t$ *is*

$$e^c_{R|N_{t+1}-N_t=0} = 0; \tag{20}$$

2. *the bond with payoff* $X^b_{t+1} = e^{\int_t^{t+1} r_s ds}$ *and return* $R^b_{t+1} = X^b_{t+1}/P^b_t$ *is*

$$e^b_{R|N_{t+1}-N_t=0} = \exp\big((1 - e^{-\alpha\gamma\nu})\lambda\big) - 1. \tag{21}$$

Proof. See Appendix A.3.4 ■

The results in Proposition 4.2 indicate that the risky bond carries pricing errors while the risky claim does *not* generate persistent pricing errors. The intuition behind this result is that in case of disasters, the SDF rises (consumption drops sharply, marginal utility rises), while the return on the claim falls sharply and would exactly offset the previous effect, hence the net effect on the EE error is zero. In contrast, the return on the bonds would not be affected instantaneously, so in periods without disasters, the average EE errors are negative (empirical estimates are shown in Table A.1, column EE errors).

Three remarks are noteworthy. First, EE error in (20) shows that rare events may not be relevant for moment conditions of particular assets, and will not generate persistent pricing errors. Here, the assets were chosen to obtain analytical expressions for EE errors. Second, based on excess returns we may still rationalize finite sample EE errors for excess returns,

$$e^c_{X|N_{t+1}-N_t=0} = -e^b_{R|N_{t+1}-N_t=0}.$$

Third, low-probability events such as bonanzas are also candidates for generating EE errors if the sample average is not the population mean, but less likely to drive our results.[11]

Given the EE errors, the root mean square error (RMSE) is calculated and interpreted as a measure of the magnitude of mispricing across assets.[12] The (true) RMSE is defined as the square root of the average squared (mean) EE errors across assets,

$$RMSE = \left((1/N)H_T(\phi)^\top H_T(\phi)\right)^{\frac{1}{2}}/T. \tag{22}$$

4.2 Identifying the source of the bias

In what follows, we illustrate the implications for the *estimated* pricing errors for endowment and production economies. As we show below, the misspecified moment condition leads to incorrect (biased) parameter estimates. While for the simple endowment economy with constant risk-free rates the GMM objective forces estimated EE errors to zero, we show why such errors may arise in an endowment economy with time-varying interest rates.

For illustration purposes, let us consider a claim on a risky asset and bond with default risk in the endowment economy. Use the definition of the estimated EE errors in (5), together with equilibrium process for consumption growth, $\ln(C_{t+1}/C_t)$, and the one period gross returns on these assets, R^c_{t+1} and R^b_{t+1}, respectively (see Appendix A.2). Then, conditional on no disasters, we obtain the following estimated EE errors

$$\begin{aligned}
\widehat{e^c_{R|N_{t+1}-N_t=0}} &= (\hat{\beta}/\beta)\exp\left((\gamma-\hat{\gamma})(\bar{\mu}-\tfrac{1}{2}\bar{\sigma}^2)-\tfrac{1}{2}((1-\gamma)^2-(1-\hat{\gamma})^2)\bar{\sigma}^2\right)\\
&\quad\times\exp\left((1-e^{(1-\gamma)\bar{\nu}})\lambda\right)-1,\\
\widehat{e^b_{R|N_{t+1}-N_t=0}} &= (\hat{\beta}/\beta)\exp\left((\gamma-\hat{\gamma})(\bar{\mu}-\tfrac{1}{2}\bar{\sigma}^2)-(\gamma^2-\hat{\gamma}^2)\tfrac{1}{2}\bar{\sigma}^2\right)\\
&\quad\times\exp\left((1-(1-(1-e^{\kappa})q)\,e^{-\gamma\bar{\nu}})\lambda\right)-1.
\end{aligned}$$

The result shows that by minimizing the empirical EE errors, the parameter estimates are typically biased for $(1-e^{(1-\gamma)\bar{\nu}})\lambda \neq 0$ and $(1-(1-(1-e^{\kappa})q)\,e^{-\gamma\bar{\nu}})\lambda \neq 0$, respectively.

Now the GMM objective is to choose parameters $\hat{\phi}$ such as to minimize the RMSE in (22). In particular, we encounter the square root of the average squared (mean) EE error

$$\widehat{RMSE} = \left((1/N)H_T(\hat{\phi})^\top H_T(\hat{\phi})\right)^{\frac{1}{2}}/T.$$

[11] Pricing errors may result from rare improvements in technology (e.g., cyclical growth as in Wälde, 2005). Hence, $e^c_{R|\bar{N}_{t+1}-\bar{N}_t=0} = \exp\left((1-e^{-\gamma\bar{\nu}})\bar{\lambda}\right)-1$, can be either positive or negative in times without innovations.

[12] To give a sense of how large pricing errors are, the RMSE is often reported relative to the returns of the assets. We define the square root of the average squared (mean) returns (RMSR) as

$$RMSR = \left((1/N)R_T^\top R_T\right)^{\frac{1}{2}}/T$$

where R_T/T is the sample average of the N-vector of asset returns $\{R_t\}$.

Therefore, the EE puzzle is not really about how to generate pricing errors, but rather how to generate *estimated* EE errors in finite samples. Consider the case $\bar{\nu} < 0$ (and $\kappa < 0$). The EE error for the risky asset (conditional on no disasters) in (16) is positive for $\gamma < 1$, whereas negative for $\gamma > 1$. For the riskless asset in (17), this EE error on average is negative in quiet times for reasonable parameterizations. The bias in the parameter estimates should eliminate such pricing errors. In fact, the estimated EE error is eliminated for

$$\begin{aligned}\hat{\gamma} &= \gamma - (e^{(1-\gamma)\bar{\nu}} - e^{-\gamma\bar{\nu}} + (1-e^{\kappa})qe^{-\gamma\bar{\nu}})\lambda/\bar{\sigma}^2, \\ \hat{\beta} &= \beta \exp\left((\gamma-\hat{\gamma})(\bar{\mu} - \tfrac{1}{2}\bar{\sigma}^2) - (\gamma^2 - \hat{\gamma}^2)\tfrac{1}{2}\bar{\sigma}^2 - ((1-(1-e^{\kappa})q)\, e^{-\gamma\bar{\nu}} - 1)\lambda\right).\end{aligned}$$

The reason for this result is that the risk premium (19) generated by disaster risk is captured by the degree of Gaussian uncertainty determined by $\bar{\sigma}^2$, and an estimate of risk aversion $\hat{\gamma}$. In this example, the (asymptotic) bias of the parameter estimate for γ amounts to

$$\gamma - \hat{\gamma} = -(e^{-\gamma\bar{\nu}}(1-e^{\bar{\nu}}) - (1-e^{\kappa})qe^{-\gamma\bar{\nu}})\lambda/\bar{\sigma}^2, \quad (23)$$

which for $q = 0$ is unambiguously negative ($\hat{\gamma} > \gamma$), while it increases in λ and decreases in $\bar{\sigma}$. Our results show that the larger the disaster risk premium, the larger the absolute bias in the estimate of γ. Moreover, the bias will be larger for $\bar{\sigma}^2$ being close to zero.

Our analytical result in (23) not only explains the bias in the parameter estimates, it also shows how the estimated EE error in models with rare events and with constant investment opportunities $\bar{\mu}$ can be eliminated (up to numerical accuracy). Intuitively, the risk premium from Gaussian uncertainty needs to account for the disaster risk premium in the data, which works through a large estimate of risk aversion $\hat{\gamma}$ for given $\bar{\sigma}$.

For changing investment opportunities μ_t, however, the GMM objective might fail to eliminate estimated EE errors. As we show below in simulations, the endowment economy mimicking a production economy with time-varying interest rate dynamics as in (14) can generate estimated EE errors, because the bias in the parameter estimates often cannot eliminate the average pricing errors for given $\bar{\sigma}^2$ in the GMM objective.

4.3 Resolving the misspecification

In this section, we discuss three potential solutions to the pricing error puzzle, when the risk of rare disasters is present. First, we may go for longer samples in which rare disasters are included.[13] Second, we may use samples without disasters and modify the moment condition. Third, we may allow for a constant term which accounts for unexplained risk premia.

[13]Note that a cross-sectional approach would not be able to match the population conditions, unless we can infer the jump probabilities (and jump sizes) from the data and correctly specify the moment conditions.

4.3.1 Long samples

Because the property we are describing is a short-sample phenomenon, it is tempting to go for longer samples. It is worth making two points about this approach. First, in cases where the longer sample does not include sufficiently large consumption disasters, this approach would not help. For example, the US did not experience consumption disasters of the magnitude used in the illustration above over the period 1900-2008 (including the Great Depression). Second, even if rare disasters were included in the sample, it would be rather a coincidence if their sample frequency matched the population frequency for the given parameterization.

To illustrate this point, with $\lambda = 0.017$ (cf. Barro, 2006), we expect 1.7 disasters in 100 years of data. Therefore, we would need exactly *one* disaster for a sample of 60 years, or 2 disasters for a sample of 120 years of data. Taking this point seriously, longer samples may help to reduce the bias (and pricing errors) but given data availability will eventually fail. Because the data generating process gives rise to peso problems, the moment conditions for the estimation of the C-CAPM are likely to be incorrectly specified.

4.3.2 Resuscitating the moment conditions

Recall that the problem with the C-CAPM estimation is that in samples with no disasters, the estimates for the parameter vector $\phi = (\beta, \gamma)^\top$ are biased because the moment conditions are not correctly specified. In such samples, we need to correct for the conditional pricing errors in (16) and (17) for the endowment economy, or in (20) and (21) for the production economy, such that the correct moment condition for the two assets reads

$$\tilde{h}_{t+1} = \begin{pmatrix} e^b_{R,t+1} - e^b_{R|N_{t+1}-N_t=0} \\ e^c_{R,t+1} - e^c_{R|N_{t+1}-N_t=0} \end{pmatrix} = \begin{pmatrix} \beta(C_{t+1}/C_t)^{-\gamma} R^b_{t+1} - 1 - e^b_{R|N_{t+1}-N_t=0} \\ \beta(C_{t+1}/C_t)^{-\gamma} R^c_{t+1} - 1 - e^c_{R|N_{t+1}-N_t=0} \end{pmatrix}, \qquad (24)$$

and thus, any remaining EE pricing error (corrected for the disaster risk) should be zero. Generally, the correction will depend on the parameter vector ϕ, which makes the adjustment dependent on the particular model (and assets) at hand. Below we report the empirical results for the endowment economy and the production economy.

This approach becomes relevant when the researcher can safely assume that consumption disasters have not occurred in the sample (henceforth 'conditional GMM'), which essentially can be interpreted as a bias correction. Restricting the attention to quiet times is not a panacea, though, as it requires a judgement based on data prior to estimation. Moreover, in long samples which include rare disasters, the moment conditions would then *not* be correctly specified and conditional GMM does not help to get unbiased estimates. Further challenges are that the correction depends on (1) the underlying model (endowment vs. production), (2) the particular asset class (bonds vs. claims), and (3) the particular calibration of parameters (arrival rate and size of disasters). To get an idea on how sensitive the results are with respect to the underlying model, we compare the results of two different models in order to get an idea on the consequences of a specific moment condition.

From the moment conditions (24), we see that this approach requires fixing some of the structural model parameters. These moment conditions depend on the particular asset under consideration. If the asset is subject to default risk, we need to account for this risk in the moment condition. One direction for research would be to use different assets that carry different risk premia to identify and estimate such parameters, including those determining the risk of disasters (the distribution, size and/or the frequency of disasters).

4.3.3 Include constants

One issue with resolving the misspecification by subtracting the known EE errors is that they are model-specific and depend on the particular asset. Alternatively, would it be possible to capture the unobserved disaster risk by adding constants? Parker and Julliard (2005) suggest including a constant (for the excess returns) in order to give the model the ability to explain the equity premium. This procedure should capture consumption risk, which underpredicts the excess returns of all assets by the *same* amount (Parker and Julliard, 2005, p.192).

Following Parker and Julliard (2005, eq. 6) for the contemporaneous effect, with just two moment conditions (neglecting default risk), we may fix μ_0 and estimate γ and α_0 using

$$\bar{h}_{t+1} = \begin{pmatrix} R^f_{t+1}(C_{t+1}/C_t)^{-\gamma}(R^c_{t+1} - R^f_{t+1})/\mu_0 - \alpha_0 \\ R^f_{t+1}(C_{t+1}/C_t)^{-\gamma} - \mu_0 \end{pmatrix}$$

Only if theoretical and sample moments were the same, the constant α_0 would be zero. Hence, the constant α_0 measures the ex-post mispricing for the excess return.

Following this idea, we may fix γ and estimate separately the disaster risk adjustments α_1 and α_2.[14] The correct moment condition accounts for the EE errors,

$$\bar{h}_{t+1} = \begin{pmatrix} e^b_{R,t+1} - \alpha_1 \\ e^c_{R,t+1} - \alpha_2 \end{pmatrix} = \begin{pmatrix} \beta(C_{t+1}/C_t)^{-\gamma}R^b_{t+1} - 1 - \alpha_1 \\ \beta(C_{t+1}/C_t)^{-\gamma}R^c_{t+1} - 1 - \alpha_2 \end{pmatrix} \tag{25}$$

or

$$\bar{h}_{t+1} = \begin{pmatrix} (C_{t+1}/C_t)^{-\gamma}R^b_{t+1} - (1+\alpha_1)/\beta \\ \beta/(1+\alpha_1)(C_{t+1}/C_t)^{-\gamma}R^c_{t+1} - 1 - \alpha_0 \end{pmatrix} \tag{26}$$

where we fix $\beta/(1+\alpha_1)$ instead and estimate γ together with $\alpha_0 = (1+\alpha_2)/(1+\alpha_1)-1$. The estimates give an empirical measure of the mispricing for the two assets under consideration for a given parameterization for β and γ. This provides some indication on whether the disaster risk underpredicts excess return by the same amount or not.

[14] Restricting the yet unknown parameters is only necessary in the two-asset case. In a larger cross-section, we estimate the disaster risk adjustments α_1 and α_2 together with β and γ (see Table A.2).

For the endowment economy, we may infer λ and $\bar{\nu}$ (suppose the default risk q is zero), conditional on a sample without consumption disasters,

$$\begin{aligned} e^{f}_{R|N_{t+1}-N_t=0} - \alpha_1 = \exp\left((1-e^{-\bar{\nu}\gamma})\lambda\right) - 1 - \alpha_1 &= 0 \\ \Leftrightarrow \ \exp\left((1-e^{-\bar{\nu}\gamma})\lambda\right) &= 1+\alpha_1 \\ \Leftrightarrow \ (1-e^{-\bar{\nu}\gamma})\lambda &= \log(1+\alpha_1) \end{aligned} \tag{27}$$

and

$$\begin{aligned} e^{c}_{R,t+1} - \alpha_2 = \exp\left((1-e^{(1-\gamma)\bar{\nu}})\lambda\right) - 1 - \alpha_2 &= 0 \\ \Leftrightarrow \ \exp\left((1-e^{(1-\gamma)\bar{\nu}})\lambda\right) &= 1+\alpha_2 \\ \Leftrightarrow \ (1-e^{(1-\gamma)\bar{\nu}})\lambda &= \log(1+\alpha_2) \end{aligned} \tag{28}$$

such that

$$(1-e^{(1-\gamma)\bar{\nu}}) - (1-e^{-\bar{\nu}\gamma}) = \log\left((1+\alpha_2)/(1+\alpha_1)\right) \approx \alpha_0 \tag{29}$$

gives an empirical measure of the disaster risk premium in (19). Below we provide estimates of the disaster risk along with the standard parameters (cf. Table A.2). If we obtained a measure of that premium from other sources (e.g., inferred from derivatives), we may subtract the sample mean of the moments in a two-step procedure.[15]

4.4 Data

In what follows we present the data for our empirical estimates of the C-CAPM parameters and show how the results are affected by peso problems in the estimation.

For financial data, we use US postwar quarterly returns (1951:Q4-2016:Q4) on a broad stock market index return (CRSP value-weighted price index return, R^m_t), and the short-term bond return from US Treasury-Bills (three-month rate, R^b_t, henceforth T-Bill).[16] We also show results when including 6 size and book-to-market Fama-French portfolios, R^{FF}_t. For a longer (and international) sample, we use the annual returns from Global Financial Data (1900-2008) for a selected set of 6 countries including the United States (US), Canada (CAN), Germany (GER), Italy (ITA), Japan (JAP), and United Kingdom (UK).[17]

For consumption, we use the real (chain-weighted) personal consumption expenditures on nondurable goods and services per capita at a quarterly frequency (1951:Q4-2016:Q4).[18] We

[15] Sönksen and Grammig (2021) propose a similar two-step simulation-based strategy to estimate rare disaster risk models.

[16] Board of Governors of the Federal Reserve System, 3-Month Treasury Bill [TB3MS], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/TB3MS.

[17] Global Financial Data: https://www.globalfinancialdata.com

[18] US Bureau of Economic Analysis, Real personal consumption expenditures per capita: Nondurable goods [A796RX0Q048SBEA], retrieved from FRED; https://fred.stlouisfed.org/series/A796RX0Q048SBEA.

use the (standard) timing convention that consumption takes place at the end of a period. For the longer (international) sample, we use consumption data from the Barro and Ursúa (2008) macroeconomic data set.[19]

4.5 Empirical results

In this section, we provide some empirical estimates of pricing errors. For conditional GMM (where we condition on a sample without disasters), we show the results for both the simple endowment and time-varying endowment economy (mimicking a production economy).

First, we confirm economically large unconditional estimated EE errors of 4.5% p.a., similar to Lettau and Ludvigson (2009), for the postwar US data (annual) from 1951-2008 (cf. Table A.1, column GMM). For the international (longer) sample we find large RMSE for CAN, GER, ITA, and JAP ranging from 4.8% to 9.3% p.a. Our empirical results also show that the model does *not* necessarily produce large *estimated* EE errors for all samples, i.e., across different time spans and/or countries. A generally more robust result is that the parameter estimates seem to be severely biased for different sample periods and across countries (consistent with the findings in the literature). The bias tends to be larger for the shorter (postwar) data relative to the longer sample.

The next 6 columns (columns conditional GMM) report the GMM estimates and estimated EE errors with the respective theoretical (model-specific) correction for the two models. Here, we resolve the empirical problems for the postwar samples, i.e., the estimated EE errors are eliminated, and the parameter estimates for both time preference and risk aversion of the canonical model are much more plausible for both the simple endowment and the time-varying endowment economy (column production). For the longer sample, two observations are remarkable. We did not expect the correction to work properly for a sample that includes disasters (the condition requires to exclude disasters), as is the case at least for GER and JAP between 1900 and 1950. For the suggested bias correction we should pick a sample without disasters to correctly adjust the sample mean. In fact, the conditional GMM estimates for GER still produce economically large EE errors of nearly 4.0% p.a.

In the next 3 columns (columns EE errors), we report the GMM estimates for the disaster risk adjustments, α_0 and α_1 and the estimated EE errors, for given parameter values for time preference and risk aversion. For the case of GER, the estimated disaster risks adjustments are orders of magnitudes higher than other estimates. This may indicate that the sample frequency of disasters was even *higher* than the population frequency in the longer sample 1900-2008. Recall that the sample includes two severe World War episodes, hyperinflation and the Great Depression with large capital stock destruction and/or consumption disasters.

[19] Barro-Ursúa Macroeconomic Data: https://scholar.harvard.edu/barro/data_sets

In the postwar period (1951-2008) where we can safely ignore severe disasters, the estimated disaster risk adjustments are more plausible, but largest for GER and JAP.

The last 3 columns (columns Parker/Julliard) report the GMM estimates for the constant α_0 and estimated EE errors assuming a production economy (fixing the rate of time preference). Based on the postwar sample, the average disaster risk premium for the T-Bill is about the order of magnitude of Barro's parameterization, ranging from 5.1% (CAN) to 11.7% (GER). For the US we get similar estimates about 6.6% for the postwar annual data.

For the larger cross-section, when we add 6 Fama-French (FF) portfolios sorted by size and book-to-market, a similar picture emerges (cf. Table A.2). Again, the standard GMM approach gives biased estimates: $\hat{\beta} = 0.84$, $\hat{\gamma} = 147.1$ with EE errors of about 1.7%. The estimated EE errors for plausible parameters of time preference and risk aversion are economically important, but differ substantially across assets. Being consistent with our theoretical result that EE depend on the specific asset at hand, it challenges the assumption that portfolio returns are underpredicted by the same amount (cf. Parker and Julliard, 2005, p.193). Given that for three FF portfolios (i.e., Big Value, Big Neutral, and Small Value) the EE errors are estimated at similar orders of magnitude, we also estimate EE errors by imposing the restriction $\alpha_3 = \alpha_4 = \alpha_6$. This restriction enables us to estimate the parameters β and γ along with the risk adjustments for the bond, the risky asset and of 4 FF portfolios. Once we account for the EE errors, we get more plausible parameter estimates: $\hat{\beta} = 0.99$, $\hat{\gamma} = 4.6$ with EE errors of 0.3%. For the endowment economy, the implied average size of disasters from (27) and (28) then is $\bar{\nu} = -0.53$ and arrival rate $\lambda = 0.015$. These estimates are in line with the empirical measures of disasters in Barro (2006).

4.6 Simulation results

In this section we investigate whether it is possible to reproduce the empirical failure of the C-CAPM by using simulated data in models with either rare events or long-run risk. After simulating the data, we estimate the parameters of a power utility C-CAPM pricing kernel whose parameters an econometrician would estimate when she is confronted with the data similar to Lettau and Ludvigson (2009). We are mainly interested in investigating whether the models generate *estimated* pricing errors using the data where we condition on no disasters. Does a misspecified pricing kernel – despite the biased estimates for time preference and risk aversion parameters – generate EE errors? Hence, we shed light on the performance of C-CAPM estimation regarding the bias and plausibility of estimated structural parameters in the presence of a 'peso problem'.

We simulate equilibrium paths for asset returns for a risky claim, R^c_{t+1}, and risky bond, R^b_{t+1}, as well as for consumption growth, $\log(C_{t+1}/C_t)$, from the parameterized consumption-based models with rare events. We consider both the simple endowment economy and the mimicking production economy presented in Section 3, for which we provide analytical expressions for asset prices (see Appendices A.2 and A.3). Consistent with the sample size in empirical studies of the C-CAPM, the simulated sample path of each of the 5,000 Monte Carlo draws has a length of 50 years. The parameterization of the models is summarized in Tables A.3 and A.4. They follow the literature on rare events in endowment and production economies, respectively (see e.g., Barro, 2009, Posch, 2009, and Wachter, 2013).[20]

In Tables A.6 to A.9 we report the results from the Monte Carlo simulations. Our main quantities of interest are the average pricing error ($RMSE$), i.e., the EE errors, the estimated EE errors ($\widehat{RMSE}$), i.e., a measure of estimated pricing errors, and the parameter estimates $\hat{\beta}$ and $\hat{\gamma}$ obtained by fitting a power utility C-CAPM to the simulated data. In addition, we report the distributional properties of asset returns, equity premium, and consumption growth (in annualized percentage terms). For ease of readability, we report results *conditional* on the case of no disasters, i.e., only for those cases in which no disaster happened to occur over the 50 years period even though such diasters were expected by the market participants ex-ante (peso problem).[21] Conditioning on no disasters is interpreted as studying a sample such as the postwar period without major consumption disasters (Barro, 2006). The simulations also include more frequent low-probability events, i.e., 'smaller' jumps in productivity, without changing our main results (cf. Tables A.8 and A.9). Hence, only rare disasters with major economic consequences rationalize large empirical EE errors.

Our results support our claim that peso problems can have a strong impact on the estimates of structural parameters and pricing errors. For example, relative to the endowment economy with zero probability of rare disasters (cf. Table A.7), the endowment economy with low probability consumption disasters in Table A.6 on average generates severely biased parameter estimates of $\hat{\beta} = 1.17$ and $\hat{\gamma} = 804.6$. While we do not find (estimated) pricing errors of the C-CAPM in the endowment economy, we find substantial pricing errors in the endowment economy mimicking a production economy. Our results in Tables A.8 and A.9 show that the C-CAPM (with power utility) generates large pricing errors on average between 2% and 2.4% (indicated by $\widehat{RMSE}$ in the tables), of similar size as the 2.5% observed in the data. Thus, departures from log-normality in the cases where we conditioned on no disasters, e.g., through to changing investment opportunities in the production economy, seem to be important to generate estimated EE errors.

[20] Julliard and Ghosh (2012) criticize the calibration of one-year contractions as being equal to the cumulated multi-year contractions recorded in the data. As illustrated in Tsai and Wachter (2015), with clustering of shocks and Epstein-Zin preferences, a rare events model still produces a sizeable equity risk premium.

[21] For illustration, the unconditional results, and simulation results for different parameterizations, and/or scenarios can be found in the Online Appendix (see Section B.3).

Our economies with rare events are parameterized to $\gamma = 4$ for the coefficient of relative risk-aversion and $\beta = 0.97$ for the subjective time discount factor. Yet, if anticipated consumption disasters do not occur in sample, we obtain the biased and implausible parameter estimates that are well-known from empirical results in the literature. Our simulation results suggest that such biased and implausible parameter estimates are not surprising in a world where agents are concerned about rare negative consumption shocks.

Finally, we repeat our experiment by simulating equilibrium paths for asset prices and consumption from the long-run risk (LRR) model with recursive preferences and estimate the standard C-CAPM (a description of the model can be found in Appendix A.4). Tables A.10 and A.11 summarize our findings when the model is parameterized as in Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012), respectively.[22] Similar to the findings reported in Lettau and Ludvigson (2009), the estimated value for β is close to its true value, while the estimate for γ is moderately biased. Moreover, the estimated EE errors are numerically zero, implying that the LRR model is unable to rationalize large pricing errors despite the model's stochastically changing investment opportunity set.

To summarize, unlike models of habit formation and/or long-run risk we refer to in the introduction, models incorporating rare events are able to solve the pricing error puzzle. Although the rare events and long-run risks (LRR) models are considered as complementary approaches (see Barro and Jin, 2021), the ability of the rare event model to additionally solve the pricing error puzzle can be used to further discriminate between the two leading theories of asset pricing. Our result thus complements the literature by adding a solution to another dimension of the puzzles. At the same time, we find a severe bias in parameter estimates of the subjective time discount factor and the coefficient of relative risk aversion for cases in which consumption disasters do not occur in the sample. Our results suggest that the Barro-Rietz rare events hypothesis together with a changing investment opportunity set is able to account for the poor performance of the C-CAPM.

5 Conclusion

In this paper we study the impact of rare events (such as wars or natural catastrophes) on Euler equation (EE) errors and the empirical performance of the consumption-based asset pricing model in general. For this purpose, we derive analytical asset pricing implications and EE errors both in an endowment as well as a production economy with stochastically occurring disasters. In extensive simulations we also investigate the impact of rare events

[22] Because the true EE error is not available analytically in the LRR model, we report pricing errors as $R^i_{t+1} - E(R^i_{t+1})$ for asset i, respectively, and replace simulated martingale increments h_t by the asset returns in deviation from the unconditional mean values, reported as RMSE* in Tables A.10 and A.11.

on estimates of structural parameters of the consumption-based CAPM and the empirical performance of the model. Thus, we seek to provide a better understanding of why the standard model fails so dramatically when fitted to the data.

Allowing for low-probability events in an otherwise standard C-CAPM helps explaining why the canonical model generates large and persistent EE errors when confronted by the data. Hence, the consumption-based CAPM with rare events qualifies as a class of models which rationalize pricing errors. Similar to Kocherlakota (1997), we argue that accounting for rare disasters in the C-CAPM produces reasonable parameter estimates and explains the pricing errors in the empirical data, which complements the statistical approach for heavy-tailed distributions with analytical results. We show analytically and through simulations, based on standard calibrations, that the poor empirical performance and implausible estimates of risk aversion and time preference are not puzzling in a world with Barro-Rietz disaster risk. We discuss different approaches of how the biased estimates can be avoided in empirical research and suggest a simple fix to the moment conditions by resolving the misspecification in samples without disasters.

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A Appendix

A.1 Computing moments

Lemma A.1 *The conditional mean of* c^{kN_s} *conditioned on the information set at time* t *is*

$$E_t\left[c^{kN_s}\right] = c^{kN_t} e^{(c^k-1)\lambda(s-t)}, \quad s > t, \quad c, k \in \mathbb{R},$$

which for integer k *denote the raw moments of* c^{N_s}.

Proof. We can trivially rewrite $c^{kN_s} = c^{kN_t} c^{(N_s-N_t)k}$. Thus, $E_t\left[c^{kN_s}\right] = c^{kN_t} E_t\left[c^{(N_s-N_t)k}\right]$. Computing this expectation requires the probability that a Poisson process jumps n times between t and s. Formally,

$$\begin{aligned} E_t\left[c^{(N_s-N_t)k}\right] &= \sum_{n=0}^{\infty} c^{kn} \frac{e^{-\lambda(s-t)}[(s-t)\lambda]^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{-(s-t)\lambda}[(s-t)c^k\lambda]^n}{n!} \\ &= e^{(s-t)(c^k-1)\lambda} \sum_{n=0}^{\infty} \frac{e^{-(s-t)\lambda-(s-t)(c^k-1)\lambda}[(s-t)c^k\lambda]^n}{n!} \\ &= e^{(s-t)(c^k-1)\lambda} \sum_{n=0}^{\infty} \frac{e^{-(s-t)c^k\lambda}[(s-t)c^k\lambda]^n}{n!} = e^{(s-t)(c^k-1)\lambda}, \end{aligned}$$

where $\frac{e^{-\lambda s}[\lambda s]^n}{n!}$ is the probability of $N_s = n$, and $\sum_{n=0}^{\infty} \frac{e^{-(s-t)c^k\lambda}[(s-t)c^k\lambda]^n}{n!} = 1$ is the probability function over the whole support of the Poisson distribution used in the last step. ■

Corollary A.2 *The unconditional mean of* c^{kN_s} *is*

$$E\left[c^{(N_s-N_t)k}\right] = e^{(c^k-1)\lambda(s-t)}, \quad s > t, \quad c, k \in \mathbb{R}.$$

A.2 Lucas' endowment economy with rare events

A.2.1 The model

Suppose that the ownership of the exogenously given output Y_t is determined at each instant in a competitive stock market. The production unit has outstanding one perfectly divisible equity share that entitles its owner to all of the unit's instantaneous output in t. Shares are traded at a competitively determined price, P_t^i, evolving according to

$$dP_t^i = \mu P_t^i dt + \sigma P_t^i dB_t + P_{t-}^i J_t dN_t, \tag{30}$$

where P_{t-}^i is the price of the asset an instant before a jump.

Following Barro (2006), we also consider a bond with default risk whose price P_t^b evolves according to

$$dP_t^b = P_t^b r dt + P_{t-}^b D_t dN_t, \quad \text{where} \quad D_t = \begin{cases} 0 & \text{with} \quad 1-q \\ \exp(\kappa) - 1 & \text{with} \quad q \end{cases} \tag{31}$$

is the default risk in case of a disaster, $\kappa < 0$ is the (degenerated) size of the default and q is the probability of default in case of a disaster. This asset can be thought of as a government treasury bill. Finally, there is a (shadow) risk-free asset with price dynamics

$$dP_t^f = P_t^f r^f dt, \tag{32}$$

in which r^f is the continuously compounded risk-free rate. Because prices fully reflect all available information, μ, σ, J_t, r, and r^f will be determined in general equilibrium.

Preferences. The economy is inhabited by a single consumer, interpreted as representative of a large number of identical consumers. Preferences are defined recursively by (8). For simplicity, we assume $\gamma = 1/\psi$, so the normalized aggregator (9) reads $f(C_t, U_t) = C_t^{1-\gamma}\left((1-\gamma)\right)^{-1} - \rho U_t$, and we obtain the same value function as in the case of time-separable utility. In this case, the consumer maximizes

$$U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \; u'' < 0 \tag{33}$$

subject to the budget constraint

$$dW_t = \left((\mu - r)\theta_t W_t + r W_t - C_t\right) dt + \theta_t \sigma W_t dB_t + ((J_t - D_t)\theta_{t-} + D_t) W_{t-} dN_t, \quad W_0 \in \mathbb{R}, \tag{34}$$

where W_t is real financial wealth, and θ_t denotes the consumer's share in the risky asset. Without loss of generality, we have assumed that there is no immediate access to the risk-free alternative and there are no dividend payments.

The consumer's problem can be alternatively formulated in terms of the *market portfolio*, with price $P_{M,t}$ evolving over time according to

$$dP_{M,t} = \mu_M P_{M,t} dt + \sigma_M P_{M,t} dB_t - \zeta_M(t_-) P_{M,t_-} dN_t, \tag{35}$$

where $\mu_M \equiv (\mu - r)\theta_t + r$, $\sigma_M \equiv \theta_t \sigma$, and $\zeta_M(t) \equiv (D_t - J_t)\theta_t - D_t$. The budget constraint then reads

$$dW_t = \left(\mu_M W_t - C_t\right) dt + \sigma_M W_t dB_t - \zeta_M(t_-) W_{t-} dN_t, \quad W_0 \in \mathbb{R}. \tag{36}$$

One can think of the original problem with budget constraint (34) as having been reduced to a simple Ramsey problem, in which we seek an optimal consumption rule given that income is generated by the uncertain yield of a (composite) asset (cf. Merton, 1973)[23].

[23] We may alternatively consider the portfolio problem and solve for the optimal portfolio weights. The derivations are available upon request from the corresponding author.

A.2.2 The Bellman equation and the Euler equation

Define the *value function* as

$$V(W_0) \equiv \max_{\{C_t\}_{t=0}^{\infty}} U_0, \quad s.t. \quad (36), \quad W_0 > 0. \tag{37}$$

Choosing the control $C_s \in \mathbb{R}_+$ at time s, the Bellman equation reads

$$\rho V(W_s) = \max_{C_s} \Big\{ u(C_s) + (\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \\ + \left(E^{\zeta}\left[V((1-\zeta_M(s))W_s)\right] - V(W_s)\right)\lambda \Big\}.$$

Hence, we obtain the *first-order condition* as

$$u'(C_t) = V_W(W_t), \tag{38}$$

for any $t \in [0, \infty)$, making consumption a function of the state variable $C_t = C(W_t)$.

It can be shown that the *Euler equation* is given by (cf. Posch, 2011)

$$\begin{aligned} du'(C_t) \quad = \quad & \Big((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t u''(C_t) C_W \\ & -E^{\zeta}\left[u'(C((1-\zeta_M(t))W_t))(1-\zeta_M(t))\lambda\right]\Big) dt \\ & -\pi_t u'(C_t) dB_t + \Big(u'(C((1-\zeta_M(t_-))W_{t-})) - u'(C(W_{t-}))\Big) dN_t, \end{aligned} \tag{39}$$

where $\pi_t \equiv -\sigma_M W_t u''(C_t) C_W / u'(C_t)$ is the market price of risk, and C_W is the marginal propensity to consume out of wealth, i.e., the slope of the *consumption function*.

Proposition A.3 (Optimal consumption-wealth ratio) *If utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \gamma$, then the optimal consumption-wealth ratio is constant, $C_t/W_t = b$, where $b \equiv \left(\rho + \lambda - (1-\gamma)\mu_M - (1-\zeta_M)^{1-\gamma}\lambda + (1-\gamma)\gamma\frac{1}{2}\sigma_M^2\right)/\gamma$.*

Proof. The proof closely follows Merton (1971) and Posch (2011). ■

Equilibrium properties. The economy is closed and all output will be consumed, $C_t = Y_t$. Market clearing implies that consumption growth rates are exogenous. Further, the risk-free asset is in zero net supply, and all financial wealth is invested in the risky asset, $\theta = 1$. In the Online Appendix B.1 we use these equilibrium properties to compute the general equilibrium prices μ_M, σ_M, and ζ_M.

A.2.3 General equilibrium consumption growth rates and asset returns

Consumption. From the dividend process (7), consumption growth rates are

$$Y_s \quad = \quad Y_t e^{(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)(s-t) + \bar{\sigma}(B_s - B_t) + \bar{\nu}(N_s - N_t)} \tag{40}$$

$$\Leftrightarrow \ln(C_s/C_t) \quad = \quad \ln(Y_s/Y_t) = (\bar{\mu} - \tfrac{1}{2}\bar{\sigma}^2)(s-t) + \bar{\sigma}(B_s - B_t) + \bar{\nu}(N_s - N_t). \tag{41}$$

Proposition A.4 (Stochastic discount factor) *If utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \gamma$, then the stochastic discount factor (SDF) is*

$$m_s/m_t = e^{-\left(r-e^{-\gamma\bar{\nu}}(1-e^{\kappa})q\lambda+\frac{1}{2}(\gamma\bar{\sigma})^2+(e^{-\bar{\nu}\gamma}-1)\lambda\right)(s-t)-\gamma\bar{\sigma}(B_s-B_t)-\gamma\bar{\nu}(N_s-N_t)}, \tag{42}$$

where $r = \rho + \gamma\bar{\mu} - \frac{1}{2}\gamma(1+\gamma)\bar{\sigma}^2 + \lambda - (1-(1-e^{\kappa})q)\, e^{-\gamma\bar{\nu}}\lambda$ is the continuously compounded equilibrium rate of return of the riskless security that is subject to default risk.

Proof. In general equilibrium, the Euler equation (39) reduces to

$$\begin{aligned} du'(C_t) &= (\rho - r)u'(C_t)dt + (1-e^{\kappa})u'(e^{\bar{\nu}}C_t)q\lambda dt - (u'(e^{\bar{\nu}}C_t) - u'(C_t))\lambda dt \\ &\quad -\pi_t u'(C_t)dB_t + (u'(e^{\bar{\nu}}C_{t-}) - u'(C_{t-}))dN_t, \end{aligned}$$

where the deterministic term consists firstly of the difference between the subjective rate of time preference and the riskless rate, secondly a term which transforms this rate into the certainty equivalent rate of return (shadow risk-free rate), and thirdly the compensation which transforms the Poisson process to a martingale.

For $s \geq t$, the stochastic discount factor is defined by

$$\begin{aligned} m_s/m_t &\equiv \exp\left(-\int_t^s \left(\rho - \frac{u''(C_v)C_v}{u'(C_v)}\bar{\mu} - \frac{1}{2}\frac{u'''(C_v)C_v^2}{u'(C_v)}\bar{\sigma}^2 + \frac{1}{2}\pi^2\right) dv\right) \\ &\quad \times \exp\left(-\int_t^s \pi_t dB_v + \int_t^s \left(\ln u'(e^{\bar{\nu}}C_{t-}) - \ln u'(C_{t-})\right) dN_v\right). \end{aligned} \tag{43}$$

For the case $-u''(C_t)C_t/u'(C_t) = \gamma$ we get $u'''(C_t)C_t^2/u'(C_t) = \gamma(1+\gamma)$. Furthermore, in general equilibrium the market price of risk is given by $\pi_t = -\gamma\bar{\sigma}$. Substitution into the expression for the SDF yields the desired result. ■

Proposition A.5 (Risk-free rate) *The instantaneous risk-free rate or the continuously compounded return to the risk-free asset is*

$$r^f = \rho + \gamma\bar{\mu} - \tfrac{1}{2}\gamma(1+\gamma)\bar{\sigma}^2 + \lambda\left(1 - e^{-\gamma\bar{\nu}}\right). \tag{44}$$

Proof. Consider an asset with unit payoff $X_{t+1}^f = 1$ for all t, such that the one period gross return is $R_{t+1}^f = 1/P_t^f$. From (2), the equilibrium price of such an asset at time t is

$$\begin{aligned} P_t^f &= E_t\left[\frac{m_s}{m_t}\right] = e^{-(\rho+\gamma\bar{\mu}-\frac{1}{2}\gamma\bar{\sigma}^2)(s-t)} E_t\left[e^{-\gamma\bar{\sigma}(B_s-B_t)}\right] E_t\left[e^{-\gamma\bar{\nu}(N_s-N_t)}\right] \\ &= e^{-(\rho+\gamma\bar{\mu}-\frac{1}{2}(1+\gamma)\bar{\sigma}^2+\lambda-e^{-\gamma\bar{\nu}}\lambda)(s-t)}, \end{aligned}$$

where we have used the definition of the SDF (42) and Lemma A.1. For any $s > t$

$$R_s^f = 1/P_t^f = e^{(\rho+\gamma\bar{\mu}-\frac{1}{2}(1+\gamma)\bar{\sigma}^2+\lambda-e^{-\gamma\bar{\nu}}\lambda)(s-t)}. \tag{45}$$

denotes the one-period holding gross return to the risk-free asset. The desired result follows by computing $r^f = \log(R_s^f)$. ■

Proposition A.6 (Riskless asset with default risk) *The one-period holding gross return of a riskless asset with payoff* $X^b_{t+1} = e^{\int_t^{t+1}\ln(1+D_s)dN_s}$ *for all* t*, where* D_s *captures default risk in case of disasters as defined in* (31) *is*

$$R^b_{t+1} = e^{r^f+(1-e^{\kappa})qe^{-\gamma\bar{\nu}}\lambda+\int_t^{t+1}\ln(1+D_v)dN_v}. \tag{46}$$

Proof. Substitute the random payoff $X_{b,t+1}$ together with the definition of the SDF (42) into (2), and make use of Lemma A.1, to compute the equilibrium price of the riskless asset subject to default risk at time t as

$$\begin{aligned} P^b_t &= E_t\left[\frac{m_{t+1}}{m_t}e^{\int_t^{t+1}\ln(1+D_s)dN_s}\right] \\ &= E_t\left[e^{-(\rho+\gamma\bar{\mu}-\frac{1}{2}\gamma\bar{\sigma}^2)-\gamma\bar{\sigma}(B_{t+1}-B_t)-\gamma\bar{\nu}(N_{t+1}-N_t)}e^{r+\int_t^{t+1}\ln(1+D_s)dN_s}\right] \\ &= e^{-(\rho+\gamma\bar{\mu}-\frac{1}{2}\gamma\bar{\sigma}^2)}E_t\left[e^{-\gamma\bar{\sigma}(B_{t+1}-B_t)}\right]E_t\left[e^{(e^{\ln(1+D_t)-\gamma\bar{\nu}}-1)\lambda}\right] \\ &= e^{-r}, \end{aligned}$$

where $r = r^f + \lambda q(1-e^{\kappa})e^{-\gamma\bar{\nu}}$ is the *face value* of the riskless security, i.e., the instantaneous return received by investors if no default occurs. For any $s > t$, $R^b_s = X_{b,s}/P^b_t$ denotes the gross return to the riskless asset that is subject to default risk. The desired result follows by setting $s = t+1$. ■

Proposition A.7 (Risky asset) *The one-period holding gross return on a claim to a one period ahead output,* $X_{c,t+1} = Y_{t+1}$*, is*

$$R^c_{t+1} = e^{\rho+\gamma\bar{\mu}-\frac{1}{2}\gamma\bar{\sigma}^2-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda+\bar{\sigma}(B_{t+1}-B_t)+\bar{\nu}(N_{t+1}-N_t)}. \tag{47}$$

Proof. Substitute the random payoff $X_{c,t+1}$ together with the definition of the SDF (42) into (2), and make use of Lemma A.1, to compute the equilibrium price of the risky claim at time t as

$$\begin{aligned} P^c_t &= E_t\left[\frac{m_s}{m_t}Y_s\right] \\ &= e^{-(\rho+(\gamma-1)\bar{\mu}+\frac{1}{2}(1-\gamma)\bar{\sigma}^2)(s-t)}E_t\left[e^{(1-\gamma)\bar{\sigma}(B_s-B_t)}\right]E_t\left[e^{(1-\gamma)\bar{\nu}(N_s-N_t)}\right]Y_t \\ &= e^{-(\rho-(1-\gamma)\bar{\mu}+\frac{1}{2}(1-\gamma)\bar{\sigma}^2-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda)(s-t)}Y_t. \end{aligned}$$

Then, the gross return for any $s > t$ is given by

$$R^c_s = \frac{Y_s}{P^c_t} = e^{(\rho+\gamma\bar{\mu}-\frac{1}{2}\gamma\bar{\sigma}^2-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda)(s-t)+\bar{\sigma}(B_s-B_t)+\bar{\nu}(N_s-N_t)}.$$

where we inserted (40). The desired result follows by setting $s = t+1$. ■

Proposition A.8 (Future dividends) *Consider a claim on the tree (ownership) which continuously pays $X_t = C_t$. Then, the one-period holding return on this asset is given by*

$$R^d_{t+1} = e^{\rho+\gamma\bar{\mu}-\frac{1}{2}\gamma\bar{\sigma}^2-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda+\bar{\sigma}(B_{t+1}-B_t)+\bar{\nu}(N_{t+1}-N_t)}. \tag{48}$$

Proof. From (2), the price of this claim is given as

$$\begin{aligned} P^d_t &= E_t\left[\int_t^\infty \frac{m_s}{m_t} C_s ds\right] \\ &= \int_t^\infty e^{-(\rho-(1-\gamma)(\bar{\mu}-\frac{1}{2}\bar{\sigma}^2)-\frac{1}{2}((1-\gamma)\bar{\sigma})^2-(1-e^{(1-\gamma)\bar{\nu}})\lambda)(s-t)} ds C_t \\ &= C_t/(\rho-(1-\gamma)(\bar{\mu}-\tfrac{1}{2}\bar{\sigma}^2)-\tfrac{1}{2}((1-\gamma)\bar{\sigma})^2-(1-e^{(1-\gamma)\bar{\nu}})\lambda) \end{aligned} \tag{49}$$

where we assumed $\rho-(1-\gamma)(\bar{\mu}-\frac{1}{2}\bar{\sigma}^2)-\frac{1}{2}((1-\gamma)\bar{\sigma})^2-(1-e^{(1-\gamma)\bar{\nu}})\lambda > 0$.

Now consider an asset which pays $X_{t+1} = P^d_{t+1} = (A_{t+1}/A_t)P^d_t$, which can be interpreted as a future on the ownership in $t+1$ (on the tree). From (2) we obtain the price of this asset in terms of the consumption good with $s = t+1$ as

$$\begin{aligned} P^D_t &= E_t\left[\frac{m_s}{m_t}(A_s/A_t)P^d_t\right] \\ &= e^{-(\rho+(\gamma-1)\bar{\mu}+\frac{1}{2}(1-\gamma)\bar{\sigma}^2)(s-t)} E_t\left[e^{(1-\gamma)\bar{\sigma}(B_s-B_t)}\right] E_t\left[e^{(1-\gamma)\bar{\nu}(N_s-N_t)}\right] P^d_t \\ &= e^{-(\rho-(1-\gamma)\bar{\mu}+\frac{1}{2}(1-\gamma)\bar{\sigma}^2-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda)(s-t)} P^d_t. \end{aligned}$$

Holding the claim on future dividends (ex dividends) we earn the price changes when selling the asset in period s, that is the (log-)difference of the price for the future dividends P^d_s and P^d_t, and the dividend payments, which is the difference of the current claim on future dividends P^d_t and the price of the next periods future dividends P^D_t (a future on the tree).[24]

$$\begin{aligned} \ln R^d_s &= \int_t^s d\ln P^d_v + \ln P^d_t - \ln P^D_t \\ &= \ln P^d_s - \ln P^D_t \\ &= \ln A_s - \ln A_t + (\rho-(1-\gamma)\bar{\mu}+\tfrac{1}{2}(1-\gamma)\bar{\sigma}^2-\tfrac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda)(s-t) \end{aligned}$$

where again $\rho-(1-\gamma)(\bar{\mu}-\frac{1}{2}\bar{\sigma}^2)-\frac{1}{2}((1-\gamma)\bar{\sigma})^2-(1-e^{(1-\gamma)\bar{\nu}})\lambda > 0$. ■

A.2.4 Proof of Proposition 4.1

Substituting the SDF (42) together with the one-period holding return (47) into (3) yields the ex-post pricing error for the risky claim

$$e^c_{R,t+1} \equiv e^{-(1-\gamma)^2\frac{1}{2}\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda+(1-\gamma)\bar{\sigma}(B_{t+1}-B_t)+(1-\gamma)\bar{\nu}(N_{t+1}-N_t)} - 1,$$

[24] Note that $\ln(P^d_t/P^D_t) = \ln(C_t/P^c_t) = C_t/P^d_t = \rho-(1-\gamma)(\bar{\mu}-\frac{1}{2}\bar{\sigma}^2)-\frac{1}{2}((1-\gamma)\bar{\sigma})^2-(1-e^{(1-\gamma)\bar{\nu}})\lambda$ can be interpreted as the expected return of dividends from t to $t+1$.

such that

$$e_R^c = E_t(e_{R,t+1}^c) = E_t\left[e^{-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda+(1-\gamma)\bar{\sigma}(B_{t+1}-B_t)+(1-\gamma)\bar{\nu}(N_{t+1}-N_t)}\right]-1$$

denotes the EE error. Conditional on no disasters, we can *rationalize* pricing errors for the risky claim

$$\begin{aligned} e^c_{R|N_{t+1}-N_t=0} &\equiv E_t(e^c_{R,t+1}|N_{t+1}-N_t=0) \\ &= E_t\left[e^{-\frac{1}{2}(1-\gamma)^2\bar{\sigma}^2-(e^{(1-\gamma)\bar{\nu}}-1)\lambda+(1-\gamma)\bar{\sigma}(B_{t+1}-B_t)}\right]-1 \\ &= \exp\left((1-e^{(1-\gamma)\bar{\nu}})\lambda\right)-1. \end{aligned}$$

Similarly, inserting the SDF together with the one-period equilibrium returns on the government bill in (46) and the risk-free asset (45), we obtain EE errors

$$\begin{aligned} e^b_R &= E_t\left[e^{e^{-\gamma\bar{\nu}}(1-e^{\kappa})q\lambda-(e^{-\bar{\nu}\gamma}-1)\lambda-\frac{1}{2}(\gamma\bar{\sigma})^2-\gamma\bar{\sigma}(B_{t+1}-B_t)-\gamma\bar{\nu}(N_{t+1}-N_t)+\int_t^{t+1}\ln(1+D_s)dN_s}\right]-1, \\ e^f_R &= E_t\left[e^{-(\frac{1}{2}(\gamma\bar{\sigma})^2+(e^{-\bar{\nu}\gamma}-1)\lambda)-\gamma\bar{\sigma}(B_{t+1}-B_t)-\gamma\bar{\nu}(N_{t+1}-N_t)}\right]-1. \end{aligned}$$

Conditional on no disasters, we can *rationalize* pricing errors

$$\begin{aligned} e^b_{R|N_{t+1}-N_t=0} &= \exp\left((1-e^{-\bar{\nu}\gamma})\lambda+e^{-\gamma\bar{\nu}}(1-e^{\kappa})q\lambda\right)-1, \\ e^f_{R|N_{t+1}-N_t=0} &= \exp\left((1-e^{-\bar{\nu}\gamma})\lambda\right)-1. \quad \blacksquare \end{aligned}$$

A.3 A production economy with rare events

A.3.1 The model

Consider a neoclassical production economy subject to rare events both in the accumulation of capital and the total factor productivity (cf. Posch, 2011).

Technology. At any time, the economy employs capital, labor, and knowledge, and these are combined to produce output. The production function exhibits constant return to scales $Y_t = A_tF(K_t, L)$, where K_t is the aggregate capital stock, L is the constant population size, and A_t is the stock of knowledge or total factor productivity (TFP), which is driven by a standard Brownian motion $\bar{B}_t$ and a Poisson process $\bar{N}_t$ with arrival rate $\bar{\lambda}$,

$$dA_t = \bar{\mu}A_tdt + \bar{\sigma}A_td\bar{B}_t + (\exp(\bar{\nu})-1)A_{t-}d\bar{N}_t, \quad A_0 \in \mathbb{R}_+. \tag{50}$$

The capital stock increases if gross investment, I_t exceeds stochastic capital depreciation,

$$dK_t = (I_t - \delta K_t)dt + \sigma K_tdZ_t + (\exp(\nu)-1)K_{t-}dN_t, \quad K_0 \in \mathbb{R}_+, \tag{51}$$

where $\delta > 0$ is the depreciation rate of capital, Z_t is a standard Brownian motion (uncorrelated with $\bar{B}_t$), and N_t is a Poisson process with constant arrival rate λ. The jump size in the capital stock is proportional and has a degenerated distribution.[25]

Preferences. The economy is inhabited by a single consumer, interpreted as representative of a large number of identical consumers. Preferences are defined recursively by (8). For simplicity, we assume $\gamma = 1/\psi$, so the normalized aggregator (9) reads $f(C_t, U_t) = C_t^{1-\gamma}\left((1-\gamma)\right)^{-1} - \rho U_t$, and we obtain the same value function as in the case of time-separable utility. In this case, the consumer maximizes

$$U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \; u'' < 0 \tag{52}$$

subject to the budget constraint

$$dW_t = ((r_t - \delta) W_t + w_t - C_t) dt + \sigma W_t dZ_t + J_t W_{t-} dN_t, \quad W_0 \in \mathbb{R}. \tag{53}$$

$W_t \equiv K_t / L$ denotes individual wealth, r_t is the rental rate of capital, and w_t is labor income. The paths of factor rewards are taken as given by the representative consumer.

A.3.2 The Bellman equation and the Euler equation

Define the *value function* as

$$V(W_0, A_0) = \max_{\{C_t\}_{t=0}^\infty} U_0 \quad s.t. \quad (50) \quad \text{and} \quad (53), \tag{54}$$

denoting the present value of expected utility along the optimal program. Hence, a necessary condition for optimality is provided by the Bellman's principle at time s

$$\rho V(W_s, A_s) = \max_{C_s} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s, A_s) \right\}.$$

Using Itô's formula yields

$$\begin{aligned} dV(W_s, A_s) &= ((r_s - \delta) W_s + w_s - C_s) V_W dt + V_W \sigma W_s dZ_s + V_A \bar{\mu} A_s dt + V_A \bar{\sigma} A_s d\bar{B}_s \\ &\quad + \tfrac{1}{2}\left(V_{AA} \bar{\sigma}^2 A_s^2 + V_{WW} \sigma^2 W_s^2\right) dt + [V(e^\nu W_{s-}, A_{s-}) - V(W_{s-}, A_{s-})] dN_t \\ &\quad + [V(W_{s-}, e^{\bar{\nu}} A_{s-}) - V(W_{s-}, A_{s-})] d\bar{N}_t. \end{aligned}$$

Using the property of stochastic integrals, we may write

$$\begin{aligned} \rho V(W_s, A_s) = \max_{C_s} \big\{ & u(c_s) + ((r_s - \delta) W_s + w_s - C_s) V_W + \tfrac{1}{2}\left(V_{AA} \bar{\sigma}^2 A_s^2 + V_{WW} \sigma^2 W_s^2\right) \\ & + V_A \bar{\mu} A_s + [V(e^\nu W_s, A_s) - V(W_s, A_s)]\lambda + [V(W_s, e^{\bar{\nu}} A_s) - V(W_s, A_s)]\bar{\lambda} \big\} \end{aligned}$$

[25] As in Cox, Ingersoll, and Ross (1985, p.366), individuals can invest in physical production indirectly through firms or directly, in effect creating their own firms. There is a market for instantaneous borrowing and lending at the interest rate $r_t = Y_K$, which is determined as part of the competitive equilibrium of the economy. There are markets for contingent claims which are all zero-supply assets in equilibrium.

for any $s \in [0, \infty)$. Hence, we obtain the *first-order condition*

$$u'(C_t) = V_W(W_t, A_t), \tag{55}$$

for any $t \in [0, \infty)$, making consumption a function of the state variables $C_t = C(W_t, A_t)$.

For the *evolution of the costate* we use the maximized Bellman equation

$$\begin{aligned} \rho V(W_t, A_t) &= u(C(W_t, A_t)) + ((r_t - \delta)W_t + w_t - C(W_t, A_t))V_W + V_A \bar{\mu} A_t \\ &+ \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right) + [V(e^{\nu}W_t, A_t) - V(W_t, A_t)]\lambda \\ &+ [V(W_t, e^{\bar{\nu}}A_t) - V(W_t, A_t)]\bar{\lambda}, \end{aligned} \tag{56}$$

where $r_t = r(W_t, A_t)$ and $w_t = w(W_t, A_t)$ follow from the firm's optimization problem, and the envelope theorem (also for the factor rewards) to compute the costate,

$$\begin{aligned} \rho V_W &= \bar{\mu} A_t V_{AW} + ((r_t - \delta)W_t + w_t - C_t)V_{WW} + (r_t - \delta)V_W + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &+ V_{WW}\sigma^2 W_t + [V_W(e^{\nu}W_t, A_t)e^{\nu} - V_W(W_t, A_t)]\lambda + [V_W(W_t, e^{\bar{\nu}}A_t) - V_W(W_t, A_t)]\bar{\lambda}. \end{aligned}$$

Collecting terms we obtain

$$\begin{aligned} (\rho - (r_t - \delta) + \lambda + \bar{\lambda})V_W &= V_{AW}\bar{\mu} A_t + ((r_t - \delta)W_t + w_t - C_t)V_{WW} \\ &+ \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &+ \sigma^2 V_{WW} W_t + V_W(e^{\nu}W_t, A_t)e^{\nu}\lambda + V_W(W_t, e^{\bar{\nu}}A_t)\bar{\lambda}. \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W &= V_{AW}\bar{\mu} A_t dt + V_{AW}\bar{\sigma} A_t d\bar{B}_t + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) dt + V_{WW}\sigma W_t dZ_t \\ &+ ((r_t - \delta)W_t + w_t - C_t)V_{WW} dt + [V_W(W_t, A_t) - V_W(W_{t-}, A_{t-})](d\bar{N}_t + dN_t) \end{aligned}$$

where inserting yields

$$\begin{aligned} dV_W &= (\rho - (r_t - \delta) + \lambda + \bar{\lambda})V_W dt - V_W(e^{\nu}W_t, A_t)e^{\nu}\lambda - V_W(W_t, e^{\bar{\nu}}A_t)\bar{\lambda} \\ &- \sigma^2 V_{WW} W_t dt + V_{AW} A_t \bar{\sigma} d\bar{B}_t + V_{WW} W_t \sigma dZ_t \\ &+ [V_W(e^{\nu}W_{t-}, A_{t-}) - V_W(W_{t-}, A_{t-})]dN_t + [V_W(W_{t-}, e^{\bar{\nu}}A_{t-}) - V_W(W_{t-}, A_{t-})]d\bar{N}_t, \end{aligned}$$

which describes the evolution of the costate variable. As a final step, we insert the first-order condition (55) to obtain the *Euler equation*

$$\begin{aligned} du'(C_t) &= (\rho - (r_t - \delta) + \lambda + \bar{\lambda})u'(C_t)dt - u'(C(e^{\nu}W_t, A_t))e^{\nu}\lambda dt - u'(C(W_t, e^{\bar{\nu}}A_t))\bar{\lambda} \\ &- \sigma^2 u''(C_t)C_W W_t dt + u''(C_t)(C_A A_t \bar{\sigma} d\bar{B}_t + C_W W_t \sigma dZ_t) \\ &+ [u'(C(W_{t-}, e^{\bar{\nu}}A_{t-})) - u'(C(W_{t-}, A_{t-}))]d\bar{N}_t \\ &+ [u'(C(e^{\nu}W_{t-}, A_{t-})) - u'(C(W_{t-}, A_{t-}))]dN_t, \end{aligned} \tag{57}$$

which implicitly determines the optimal consumption path.

We obtain analytical solutions for optimal consumption and asset returns only for specific parameter restrictions and particular assets. In what follows we obtain conditions under which the *policy function* $C_t = C(A_t, W_t)$, also referred to as the consumption function, is available analytically, and our variables of interest can be solved in closed form.[26]

Proposition A.9 (Optimal consumption) *Suppose the production function $F(K_t, L)$ is $Y_t = A_t K_t^\alpha L^{1-\alpha}$, utility has constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \gamma$, and the subjective discount rate $\rho = \bar{\rho}$. Then optimal consumption is proportional to income.*

$$\rho = \bar{\rho} \quad \Rightarrow \quad C_t = C(W_t, A_t) = (1-s)A_t W_t^\alpha, \quad \gamma > 1 \tag{58}$$
$$\bar{\rho} \equiv (e^{-\gamma\bar{\nu}} - 1)\bar{\lambda} + (e^{(1-\alpha\gamma)\nu} - 1)\lambda - \gamma\bar{\mu} + \tfrac{1}{2}\left(\gamma(1+\gamma)\bar{\sigma}^2 - \alpha\gamma(1-\alpha\gamma)\sigma^2\right) - (1-\alpha\gamma)\delta,$$

where $1 - s \equiv (\gamma - 1)/\gamma$ is the marginal propensity to consume.

Proof. The idea of this proof is to show that using an educated guess of the value function, the maximized Bellman equation (56) and the first-order condition (55) are both fulfilled. We guess that the value function reads

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\alpha\gamma}}{1-\alpha\gamma} A_t^{-\gamma}. \tag{59}$$

From (55), optimal consumption is a constant fraction of income, $C_t^{-\gamma} = \mathbb{C}_1 W_t^{-\alpha\gamma} A_t^{-\gamma}$ or equivalently $C_t = \mathbb{C}_1^{-1/\gamma} W_t^\alpha A_t$. Now use the maximized Bellman equation (56), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^\alpha L^{-\alpha}$, together with the transformation $K_t \equiv L W_t$, and insert the solution candidate,

$$\begin{aligned}
\rho V(W_t, A_t) \quad = \quad & \frac{\mathbb{C}_1^{-\frac{1-\gamma}{\gamma}} W_t^{\alpha-\alpha\gamma} A_t^{1-\gamma}}{1-\gamma} + ((r_t - \delta)W_t + w_t - C(W_t, A_t))V_W + V_A \bar{\mu} A_t \\
& + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right) + [V(e^\nu W_t, A_t) - V(W_t, A_t)]\lambda \\
& + [V(W_t, e^{\bar{\nu}} A_t) - V(W_t, A_t)]\bar{\lambda}.
\end{aligned}$$

Inserting the guess and collecting terms which is equivalent to

$$\begin{aligned}
(\rho - (e^{(1-\alpha\gamma)\nu} - 1)\lambda - (e^{-\gamma\bar{\nu}} - 1)\bar{\lambda}) & \frac{\mathbb{C}_1 W_t^{1-\alpha\gamma}}{1-\alpha\gamma} A_t^{-\gamma} = \frac{\mathbb{C}_1^{-\frac{1-\gamma}{\gamma}} W_t^{\alpha-\alpha\gamma} A_t^{1-\gamma}}{1-\gamma} \\
- \gamma \frac{\mathbb{C}_1 W_t^{1-\alpha\gamma}}{1-\alpha\gamma} \bar{\mu} A_t^{-\gamma} + & \left(\alpha A_t W_t^\alpha - \delta W_t + (1-\alpha) A_t W_t^\alpha - \mathbb{C}_1^{-1/\gamma} W_t^\alpha A_t\right) \mathbb{C}_1 W_t^{-\alpha\gamma} A_t^{-\gamma} \\
& + \tfrac{1}{2}\left(\gamma(1+\gamma)\bar{\sigma}^2 - \alpha\gamma(1-\alpha\gamma)\sigma^2\right) \frac{\mathbb{C}_1 W_t^{1-\alpha\gamma}}{1-\alpha\gamma} A_t^{-\gamma}.
\end{aligned}$$

[26] Similar to the endowment economy we get analytical expressions for the SDF, equilibrium consumption and asset returns for parametric restrictions $\alpha = \gamma$ with a constant consumption-wealth ratio.

Collecting terms gives

$$\rho + \gamma\bar{\mu} - \tfrac{1}{2}\left(\gamma(1+\gamma)\bar{\sigma}^2 - \alpha\gamma(1-\alpha\gamma)\sigma^2\right) + (1-\alpha\gamma)\delta \\ - (e^{(1-\alpha\gamma)\nu} - 1)\lambda - (e^{-\gamma\bar{\nu}} - 1)\bar{\lambda} = \left(\frac{\gamma}{1-\gamma}\mathbb{C}_1^{-1/\gamma} + 1\right)(1-\alpha\gamma)A_t W_t^{\alpha-1},$$

which has a solution for $\mathbb{C}_1^{-1/\gamma} = (\gamma-1)/\gamma$ and

$$\rho = (e^{-\gamma\bar{\nu}} - 1)\bar{\lambda} + (e^{(1-\alpha\gamma)\nu} - 1)\lambda - \gamma\bar{\mu} + \tfrac{1}{2}\left(\gamma(1+\gamma)\bar{\sigma}^2 - \alpha\gamma(1-\alpha\gamma)\sigma^2\right) - (1-\alpha\gamma)\delta.$$

This proves that the guess (59) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal policy function for consumption. ■

Compared to the simple endowment economy, the production economy introduces richer dynamics, which imply that consumption growth rates are endogenous and will depend on the specific solution. The dynamics will follow mainly from the marginal product of physical capital, which for parametric restrictions are given in the following result.

Proposition A.10 (Rental rate of capital) *Suppose the production function $F(K_t, L)$ is $Y_t = A_t K_t^\alpha L^{1-\alpha}$. The rental rate of capital is the marginal product of capital, $r_t = \alpha A_t K_t^{\alpha-1}$, and follows the reducible stochastic differential equation,*

$$\begin{aligned} dr_t &= c_1\big(c_2 - r_t\big)r_t dt + (\alpha-1)\sigma r_t dZ_t + \bar{\sigma} r_t d\bar{B}_t + (\exp((\alpha-1)\nu) - 1)r_{t-}dN_t \\ &\quad + (\exp(\bar{\nu}) - 1)r_{t-}d\bar{N}_t \end{aligned} \tag{60}$$

in which the constants c_1 and c_2 for the parametric restriction $\rho = \bar{\rho}$ are given by

$$c_1 \equiv \tfrac{1-\alpha}{\alpha\gamma}, \quad c_2 \equiv \alpha\gamma\delta - \tfrac{1}{2}\alpha\gamma(\alpha-2)\sigma^2 - \tfrac{\alpha\gamma}{\alpha-1}\bar{\mu}.$$

Proof. An application of Itô's lemma to the rental rate of capital, $r_t = \alpha A_t K_t^{\alpha-1}$, yields

$$\begin{aligned} dr_t &= (\alpha-1)A_t K_t^{\alpha-2}(Y_t - C_t - \delta K_t)dt + (\alpha-1)\sigma A_t K_t K_t^{\alpha-2} dZ_t \\ &\quad + (A_t K_t^{\alpha-1} - A_{t-}K_{t-}^{\alpha-1})(dN_t + d\bar{N}_t) + \tfrac{1}{2}(\alpha-1)(\alpha-2)K_t^{\alpha-3}\sigma^2 K_t^2 A_t dt \\ &\quad + K_t^{\alpha-1}(dA_t - (\exp(\bar{\nu}) - 1)A_{t-}d\bar{N}_t) \\ &= (\alpha-1)(Y_t/K_t - C_t/K_t - \delta)A_t K_t^{\alpha-1}dt + (\alpha-1)\sigma A_t K_t^{\alpha-1} dZ_t \\ &\quad + (\exp((\alpha-1)\nu) - 1)A_{t-}K_{t-}^{\alpha-1}dN_t + \tfrac{1}{2}(\alpha-1)(\alpha-2)A_t K_t^{\alpha-1}\sigma^2 dt \\ &\quad + \bar{\mu}A_t K_t^{\alpha-1}dt + \bar{\sigma}A_t K_t^{\alpha-1}d\bar{B}_t + (\exp(\bar{\nu}) - 1)A_{t-}K_{t-}^{\alpha-1}d\bar{N}_t \\ &= \tfrac{1-\alpha}{\alpha}\big(\alpha C_t/K_t + \alpha\delta - \tfrac{1}{2}\alpha(\alpha-2)\sigma^2 - \tfrac{\alpha}{\alpha-1}\bar{\mu} - r_t\big)r_t dt + (\alpha-1)\sigma r_t dZ_t + \bar{\sigma} r_t d\bar{B}_t \\ &\quad + (\exp((\alpha-1)\nu) - 1)r_{t-}dN_t + (\exp(\bar{\nu}) - 1)r_{t-}d\bar{N}_t, \end{aligned}$$

For $\rho = \bar{\rho}$ we obtain

$$\begin{aligned} dr_t &= \tfrac{1-\alpha}{\alpha}\big(\alpha\delta - \tfrac{1}{2}\alpha(\alpha-2)\sigma^2 - \tfrac{\alpha}{\alpha-1}\bar{\mu} - sr_t\big)r_t dt + (\alpha-1)\sigma r_t dZ_t + \bar{\sigma} r_t d\bar{B}_t \\ &\quad +(\exp((\alpha-1)\nu)-1)r_{t-}dN_t + (\exp(\bar{\nu})-1)r_{t-}d\bar{N}_t \\ &\equiv c_1\big(c_2 - r_t\big)r_t dt + (\alpha-1)\sigma r_t dZ_t + \bar{\sigma} r_t d\bar{B}_t + (\exp((\alpha-1)\nu)-1)r_{t-}dN_t \\ &\quad +(\exp(\bar{\nu})-1)r_{t-}d\bar{N}_t, \end{aligned}$$

which is a reducible stochastic differential equation, in which we defined $c_1 \equiv \frac{1-\alpha}{\alpha\gamma}$ and $c_2 \equiv \alpha\gamma\delta - \frac{1}{2}\alpha\gamma(\alpha-2)\sigma^2 - \frac{\alpha\gamma}{\alpha-1}\bar{\mu}$.

Because the SDE for r_t is reducible, it has the solution

$$r_s = \Theta_{s,t}\left(r_t^{-1} + c_1\int_t^s \Theta_{v,t}dv\right)^{-1}, \tag{61}$$

where $\Theta_{s,t} \equiv e^{(c_1c_2 - \frac{1}{2}((\alpha-1)\sigma)^2 - \frac{1}{2}\bar{\sigma}^2)(s-t) + (Z_s - Z_t)(\alpha-1)\sigma + (\bar{B}_s - \bar{B}_t)\bar{\sigma} + (\alpha-1)\nu(N_s-N_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t)}$. Observe that the closed-form solution enormously simplifies the problem of simulating EE errors. ■

A.3.3 General equilibrium consumption growth rates and asset returns

Consumption. Observe that the solution to (50) is for $s \geq t$

$$\begin{aligned} A_s &= A_t e^{(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)(s-t) + \bar{\sigma}(\bar{B}_s - \bar{B}_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t)} \\ \Leftrightarrow \ln(A_s/A_t) &= (\bar{\mu} - \tfrac{1}{2}\bar{\sigma}^2)(s-t) + \bar{\sigma}(\bar{B}_s - \bar{B}_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t). \end{aligned} \tag{62}$$

Similarly, we obtain growth rates of the capital stock from (51)

$$\ln(K_s/K_t) = \int_t^s \big(r_v/\alpha - C_v/K_v - \delta - \tfrac{1}{2}\sigma^2\big)dv + \sigma(Z_s - Z_t) + \nu(N_s - N_t). \tag{63}$$

For the case of $\rho = \bar{\rho}$, as from Proposition A.9, consumption is a constant fraction of output, $C_t = (1-s)Y_t$, and thus we obtain the consumption growth rate as $\ln(C_s/C_t) = \ln(Y_s/Y_t) = \ln(A_s/A_t) + \alpha\ln(K_s/K_t)$, which finally gives

$$\begin{aligned} \ln(C_s/C_t) &= 1/\gamma\int_t^s r_v dv + (\bar{\mu} - \tfrac{1}{2}\bar{\sigma}^2 - \alpha\delta - \tfrac{1}{2}\alpha\sigma^2)(s-t) + \bar{\sigma}(\bar{B}_s - \bar{B}_t) \\ &\quad +\alpha\sigma(Z_s - Z_t) + \alpha\nu(N_s - N_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t), \end{aligned} \tag{64}$$

which is endogenously determined in the production economy.

Proposition A.11 (Stochastic discount factor) *Following the assumptions in Proposition A.9, the stochastic discount factor (SDF) is given by*

$$\begin{aligned} m_s/m_t &= e^{-\int_t^s(r_v-\delta)dv + [(1-e^{(1-\alpha\gamma)\nu})\lambda + (1-e^{-\gamma\bar{\nu}})\bar{\lambda} + \gamma\alpha\sigma^2 - \frac{1}{2}(\gamma\bar{\sigma})^2 - \frac{1}{2}(\alpha\gamma\sigma)^2](s-t)} \\ &\quad \times e^{-\gamma\bar{\sigma}(\bar{B}_s - \bar{B}_t) - \alpha\gamma\sigma(Z_s - Z_t) - \alpha\gamma\nu(N_s - N_t) - \gamma\bar{\nu}(\bar{N}_s - \bar{N}_t)}. \end{aligned} \tag{65}$$

Proof. From the Euler equation (57), we obtain for $s \geq t$,

$$m_s/m_t = \exp\Bigg(-\int_t^s \left(r_l - \delta - \lambda - \bar{\lambda} + \frac{u'(C(e^{\nu}W_l, A_l))}{u'(C(W_l, A_l))} e^{\nu}\lambda + \frac{u'(C(W_l, e^{\bar{\nu}}A_l))}{u'(C(W_l, A_l))}\bar{\lambda} \right) dl$$
$$+ \frac{u''(C_l)C_W W_l}{u'(C_l)}\sigma^2 dl - \tfrac{1}{2}\int_t^s \frac{(u''(C_l))^2}{(u'(C_l))^2}((C_A A_l \bar{\sigma})^2 + (C_W W_l \sigma)^2)dl$$
$$+ \int_t^s \frac{u''(C_l)}{u'(C_l)}(C_A A_l \bar{\sigma} d\bar{B}_l + C_W W_l \sigma dZ_l)$$
$$+ \int_t^s \ln\left(\frac{u'(C(e^{\nu}W_{l-}, A_{l-}))}{u'(C(W_{l-}, A_{l-}))}\right) dN_l + \int_t^s \ln\left(\frac{u'(C(W_{l-}, e^{\bar{\nu}}A_{l-}))}{u'(C(W_{l-}, A_{l-}))}\right) d\bar{N}_l \Bigg)$$

which after inserting the policy function $C_t = C(W_t, A_t)$ gives the desired results. ■

Proposition A.12 (Risky bond) *Consider a risky asset that pays at the rate r_t in $t+1$. The one-period holding gross return of an asset with the random payoff $X_{b,t+1} = e^{\int_t^{t+1} r_s ds}$ is*

$$R^b_{t+1} \;=\; \exp\left(\int_t^s (r_v - \delta - \gamma\alpha\sigma^2 - e^{-\alpha\gamma\nu}(1 - e^{\nu})\lambda)dv\right). \tag{66}$$

Proof. Substitute the random payoff X^b_{t+1} in (2) to obtain the equilibrium price of this risky bond at time t as

$$P^b_t = E_t\left[\frac{m_{t+1}}{m_t} e^{\int_t^{t+1} r_s ds}\right].$$

Using the definition of the SDF (65) and making use of Lemma (A.1) yields

$$P^b_t = e^{\delta + \gamma\alpha\sigma^2 + e^{-\alpha\gamma\nu}\lambda - e^{(1-\alpha\gamma)\nu}\lambda}.$$

For any $s > t$, $R^b_s = X^b_s/P^b_t$ denotes the gross return on the risky bond. The desired result follows by setting $s = t+1$. ■

Proposition A.13 (Risky asset) *The one-period holding return on an asset that pays one unit of next period's capital, $X^c_{t+1} = K^{\alpha\gamma}_{t+1}$, is*

$$R^c_{t+1} = \exp\left(\int_t^{t+1} (r_v - \delta - \lambda + e^{(1-\alpha\gamma)\nu}\lambda - \gamma\alpha\sigma^2 + \tfrac{1}{2}(\alpha\gamma\sigma)^2)dv\right)$$
$$\times \exp\left(\alpha\gamma\sigma(Z_{t+1} - Z_t) + \alpha\gamma\nu(N_{t+1} - N_t)\right). \tag{67}$$

Proof. Note that for any $s > t$ it follows from (63) that

$$K^{\alpha\gamma}_s \;=\; K^{\alpha\gamma}_t e^{\int_t^s (\gamma r_v - \alpha\gamma C_v/K_v - \alpha\gamma\delta - \frac{1}{2}\alpha\gamma\sigma^2)dv + \alpha\gamma\sigma(Z_s - Z_t) + \alpha\gamma\nu(N_s - N_t)}.$$

Set $s = t+1$ and substitute the random payoff X^c_{t+1} together with the definition of the SDF (65) into (2). Making use of Lemma (A.1) compute the equilibrium price of this risky asset at time t as

$$\begin{aligned}
P^c_t &= E_t\left[\frac{m_{t+1}}{m_t}K^{\alpha\gamma}_{t+1}\right] \\
\Rightarrow \quad P^c_t &= K^{\alpha\gamma}_t e^{-(\alpha\gamma\delta+\frac{1}{2}\alpha\gamma\sigma^2-\delta-(1-e^{(1-\alpha\gamma)\nu})\lambda-(1-e^{-\gamma\bar{\nu}})\bar{\lambda}-\gamma\alpha\sigma^2+\frac{1}{2}(\gamma\bar{\sigma})^2+\frac{1}{2}(\alpha\gamma\sigma)^2)} \\
&\quad \times E_t\left[e^{-\gamma\bar{\sigma}(\bar{B}_{t+1}-\bar{B}_t)-\gamma\bar{\nu}(\bar{N}_{t+1}-\bar{N}_t)}\right] \\
&= K^{\alpha\gamma}_t e^{-(\alpha\gamma\delta+\frac{1}{2}\alpha\gamma\sigma^2-\delta-(1-e^{(1-\alpha\gamma)\nu})\lambda-\gamma\alpha\sigma^2+\frac{1}{2}(\alpha\gamma\sigma)^2)}.
\end{aligned}$$

For any $s > t$, $R^c_s = X^c_s/P^c_t$ denotes the gross return on the risky asset. The desired result follows by setting $s = t+1$. ■

Note that the risky asset considered does *not* represent a market portfolio, but is simply used to illustrate the possibility of generating Euler equation (EE) errors for particular assets.

A.3.4 Proof of Proposition 4.2

Substituting the SDF (65) together with the one-period holding return (66) into (3) yields the unconditional pricing error for the risky bond

$$\begin{aligned}
e^b_R &= E_t\left(e^b_{R,t+1}\right) \\
&= E_t\Big[e^{(1-e^{-\alpha\gamma\nu})\lambda+(1-e^{-\gamma\bar{\nu}})\bar{\lambda}-\frac{1}{2}(\gamma\bar{\sigma})^2-\frac{1}{2}(\alpha\gamma\sigma)^2} \\
&\qquad \times e^{-\gamma\bar{\sigma}(\bar{B}_{t+1}-\bar{B}_t)-\alpha\gamma\sigma(Z_{t+1}-Z_t)-\alpha\gamma\nu(N_{t+1}-N_t)-\gamma\bar{\nu}(\bar{N}_{t+1}-\bar{N}_t)}\Big] - 1.
\end{aligned}$$

Conditional on no disasters, we can *rationalize* Euler equation errors for the risky bond

$$\begin{aligned}
e^b_{R|N_{t+1}-N_t=0} &\equiv E_t\left(e^b_{R,t+1}|N_{t+1}-N_t=0\right) \\
&= \exp\left((1-e^{-\alpha\gamma\nu})\lambda\right) - 1.
\end{aligned}$$

Similarly, inserting the SDF together with one-period holding return of the risky asset (67) we obtain the EE error

$$e^c_R = E_t\left[e^{-\frac{1}{2}(\gamma\bar{\sigma})^2-(e^{-\gamma\bar{\nu}}-1)\bar{\lambda}-\gamma\bar{\sigma}(\bar{B}_{t+1}-\bar{B}_t)-\gamma\bar{\nu}(\bar{N}_{t+1}-\bar{N}_t)}\right] - 1 \quad ■.$$

A.4 The long-run risk model

A.4.1 The model

Consider an endowment economy where production, Y_t, is exogenous as in Lucas (1978). No resources are utilized, and there is no possibility of affecting the perishable output.

Technology. Following Bansal and Yaron (2004), the law of motion of Y_t is given by

$$dY_t = \mu_t Y_t dt + \sqrt{\vartheta_t} Y_t dB_t, \tag{68}$$

where μ_t is the long-run risk in the endowment, assumed to follow the square-root process

$$d\mu_t = \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu \sqrt{\vartheta_t} dB_{\mu,t}, \tag{69}$$

with persistence κ_μ, and volatility $\nu_\mu\sqrt{\vartheta_t}$. The parameter ν_μ is the volatility leverage ratio for long-run risks. Moreover, the variance ϑ_t is assumed to follow the square-root process

$$d\vartheta_t = \kappa_\vartheta(\bar{\vartheta} - \vartheta_t)dt + \nu_\vartheta \sqrt{\vartheta_t} dB_{\vartheta,t}, \tag{70}$$

with persistence κ_ϑ (see Heston, 1993). The processes $B_t, B_{\mu,t}$ and B_ϑ denote standard and independent Brownian motions. In what follows we define $d\mathbf{B}_t \equiv [dB_t\,, dB_{\mu,t}\,, dB_{\vartheta,t}]^\top$.

Ownership of any produced output is determined at each instant of time in a competitive stock market where equity shares entitle their owners to all of the future dividends. Shares are traded at a price P_t^d, and their total return evolves according to

$$dR_t^d = \mu_{R,t}dt + \boldsymbol{\sigma}_{R,t} d\mathbf{B}_t, \tag{71}$$

with $R_t^d = (dP_t^d + C_t)/P_t^d$ the return cum-dividend. The capital market also trades a risk-free asset (in zero net supply) with return r_t^f. The price of the riskless asset evolves according to

$$dP_t^f = P_t^f r_t^f dt, \quad P_0 = 1. \tag{72}$$

Preferences. Consider an economy with a single consumer, interpreted as representative of a large number of identical consumers. The consumer maximizes expected lifetime utility given by (8) and (9) subject to

$$dW_t = \left[(\mu_{R,t} - r_t^f)\theta_t W_t + r_t^f W_t - C_t\right] dt + \theta_t \boldsymbol{\sigma}_{R,t} W_t d\mathbf{B}_t, \quad W_0 \in \mathbb{R} \tag{73}$$

where r_t^f is the risk-free rate, $\mu_{R,t}$ the expected return on the risky asset, $\boldsymbol{\sigma}_{R,t} \equiv [\sigma_{R,t}^{(Y)}\,, \sigma_{R,t}^{(\mu)}, \sigma_{R,t}^{(\vartheta)}]$ its volatility, and θ_t denotes the fraction of financial wealth, W_t, invested by the consumer in the risky asset.

A.4.2 The Bellman equation

Define the *value function* as

$$V(W_0, \mu_0, \vartheta_0) \equiv \max_{\{C_t,\theta_t\}_{t=0}^\infty} U_0, \quad s.t. \quad (73), \quad W_0 > 0,\ \mu_0 \in \mathbb{R},,\ \vartheta_0 \in \mathbb{R}, \tag{74}$$

denoting the present value of expected utility along the optimal program. Hence, a necessary condition for optimality is provided by the Bellman's principle at time s

$$0 = \max_{C_s,\theta_s} \left\{ f(C_s, V(W_s,\mu_s,\vartheta_s)) + \frac{1}{dt} E_s dV(W_s,\mu_s,\vartheta_s) \right\}. \tag{75}$$

Using Itô's formula yields

$$\begin{aligned} dV(W_t,\mu_t,\vartheta_t) \;=\; & \Big(\big((\mu_{R,t} - r_t^f)\theta_t W_t + r_t^f W_t - C_t \big) V_W + \kappa_\mu(\bar{\mu} - \mu_t) V_\mu + \kappa_\vartheta(\bar{\vartheta} - \vartheta_t) V_\vartheta \\ & + \tfrac{1}{2}\theta_t^2 \boldsymbol{\sigma}_{R,t} \boldsymbol{\sigma}_{R,t}^\top W_t^2 V_{WW} + \tfrac{1}{2}\nu_\mu^2 \vartheta_t V_{\mu\mu} + \tfrac{1}{2}\nu_\vartheta^2 \vartheta_t V_{\vartheta\vartheta} \\ & + \theta_t \sigma_{R,t}^{(\mu)} W_t \nu_\mu \sqrt{\vartheta_t} V_{W\mu} + \theta_t \sigma_{R,t}^{(\vartheta)} W_t \nu_\vartheta \sqrt{\vartheta_t} V_{W\vartheta} \Big) dt \\ & + \theta_t \boldsymbol{\sigma}_{R,t} W_t V_W d\mathbf{B}_t + \nu_\mu \sqrt{\vartheta_t} V_\mu dB_{\mu,t} + \nu_\vartheta \sqrt{\vartheta_t} V_\vartheta dB_{\vartheta,t}. \end{aligned}$$

Using the properties of stochastic integrals it follows that

$$\begin{aligned} \frac{1}{dt} E_t dV(W_t,\mu_t,\vartheta_t) = & \Big((\mu_{R,t} - r_t^f)\theta_t W_t + r_t^f W_t - C_t \Big) V_W + \kappa_\mu(\bar{\mu} - \mu_t) V_\mu \\ & + \kappa_\vartheta(\bar{\vartheta} - \vartheta_t) V_\vartheta + \tfrac{1}{2}\theta_t^2 \boldsymbol{\sigma}_{R,t} \boldsymbol{\sigma}_{R,t}^\top W_t^2 V_{WW} + \tfrac{1}{2}\nu_\mu^2 \vartheta_t V_{\mu\mu} + \tfrac{1}{2}\nu_\vartheta^2 \vartheta_t V_{\vartheta\vartheta} \\ & + \theta_t \sigma_{R,t}^{(\mu)} W_t \nu_\mu \sqrt{\vartheta_t} V_{W\mu} + \theta_t \sigma_{R,t}^{(\vartheta)} W_t \nu_\vartheta \sqrt{\vartheta_t} V_{W\vartheta}, \end{aligned}$$

which substituted into (75) yields for any $t \geq 0$

$$\begin{aligned} 0 = \max_{C_t,\theta_t} \Big\{ & f(C_t, V(W_t,\mu_t,\vartheta_t)) + \big((\mu_{R,t} - r_t^f)\theta_t W_t + r_t^f W_t - C_t \big) V_W + \kappa_\mu(\bar{\mu} - \mu_t) V_\mu \\ & + \kappa_\vartheta(\bar{\vartheta} - \vartheta_t) V_\vartheta + \tfrac{1}{2}\theta_t^2 \boldsymbol{\sigma}_{R,t} \boldsymbol{\sigma}_{R,t}^\top W_t^2 V_{WW} + \tfrac{1}{2}\nu_\mu^2 \vartheta_t V_{\mu\mu} + \tfrac{1}{2}\nu_\vartheta^2 \vartheta_t V_{\vartheta\vartheta} \\ & + \theta_t \sigma_{R,t}^{(\mu)} W_t \nu_\mu \sqrt{\vartheta_t} V_{W\mu} + \theta_t \sigma_{R,t}^{(\vartheta)} W_t \nu_\vartheta \sqrt{\vartheta_t} V_{W\vartheta} \Big\}. \end{aligned} \tag{76}$$

Hence, the first-order conditions for any interior solution are

$$f_C(C_t, V) = V_W \tag{77}$$

$$\theta_t = -\frac{\mu_{R,t} - r_t^f}{\boldsymbol{\sigma}_{R,t}\boldsymbol{\sigma}_{R,t}^\top} \frac{V_W}{W_t V_{WW}} - \frac{\sigma_{R,t}^{(\mu)} \nu_\mu \sqrt{\vartheta_t}}{\boldsymbol{\sigma}_{R,t}\boldsymbol{\sigma}_{R,t}^\top} \frac{V_{W\mu}}{W_t V_{WW}} - \frac{\sigma_{R,t}^{(\vartheta)} \nu_\vartheta \sqrt{\vartheta_t}}{\boldsymbol{\sigma}_{R,t}\boldsymbol{\sigma}_{R,t}^\top} \frac{V_{W\vartheta}}{W_t V_{WW}}. \tag{78}$$

Equilibrium properties. The economy is closed and all output will be consumed, $C_t = Y_t$. Market clearing implies that the consumption growth rates are exogenous. The risk-free asset is in zero net supply and all financial wealth is invested in the risky asset, $\theta_t = 1$.

A.4.3 General equilibrium consumption growth rates and asset returns

Consumption. From the dividend process (68), consumption growth rates are given by

$$\ln(C_s/C_t) = \ln(Y_s/Y_t) = \int_t^s (\mu_v - \tfrac{1}{2}\vartheta_v) dv + \int_t^s \vartheta_v dB_v, \tag{79}$$

where μ_t and ϑ_t are the solutions to the stochastic differential equations in (69) and (70).

Proposition A.14 (Equilibrium value function) *If the preferences of the consumer are defined recursively by* (8) *and* (9)*, then for* $\psi \neq 1$ *the equilibrium value function is*

$$V(W_t, \mu_t, \vartheta_t) = \frac{W_t^{1-\gamma}}{1-\gamma} H(\mu_t, \vartheta_t), \tag{80}$$

where the function $H(\mu_t, \vartheta_t)$ *satisfies the partial differential equation*

$$\begin{aligned} 0 = \frac{1-\gamma}{1-1/\psi} \left(H(\mu_t, \vartheta_t)^{\frac{1-\psi}{1-\gamma}} - \rho \right) + (1-\gamma)\mu_t - \tfrac{1}{2}\gamma(1-\gamma)\vartheta_t \\ - \tfrac{1}{2}\psi(1-\psi)\nu_\mu^2 \vartheta_t \left(\frac{H_\mu(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} \right)^2 + \psi \kappa_\mu (\bar{\mu} - \mu_t) \frac{H_\mu(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} \\ + \tfrac{1}{2}\psi \nu_\mu^2 \vartheta_t \frac{H_{\mu\mu}(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} - \tfrac{1}{2}\psi(1-\psi)\nu_\vartheta^2 \vartheta_t \left(\frac{H_\vartheta(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} \right)^2 \\ + \psi \kappa_\vartheta (\bar{\vartheta} - \vartheta_t) \frac{H_\vartheta(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} + \tfrac{1}{2}\psi \nu_\vartheta^2 \vartheta_t \frac{H_{\vartheta\vartheta}(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)}. \end{aligned} \tag{81}$$

Proof. Conjecture that the value function takes the form in (80). Then, the normalized aggregator in (9) can be written as

$$f(C_t, V) = \frac{1}{1-1/\psi} W_t^{1-\gamma} H(\mu_t, \vartheta_t) \left((W_t/C_t)^{-(1-1/\psi)} H(\mu_t, \vartheta_t)^{\frac{1-\psi}{(1-\gamma)\psi}} - \rho \right).$$

Dividing throughout by V and substituting (90) yields

$$\frac{f(C_t, V)}{V} = \frac{1-\gamma}{1-1/\psi} \left(H(\mu_t, \vartheta_t)^{\frac{1-\psi}{1-\gamma}} - \rho \right). \tag{82}$$

Using the equilibrium properties $C_t = Y_t$ and $\theta_t = 1$, equation (76) reduces to

$$\begin{aligned} 0 = f(C_t, V) + (\mu_{R,t} W_t - C_t) V_W + \kappa_\mu (\bar{\mu} - \mu_t) V_\mu + \kappa_\vartheta (\bar{\vartheta} - \vartheta_t) V_\vartheta \\ + \tfrac{1}{2} \boldsymbol{\sigma}_{R,t} \boldsymbol{\sigma}_{R,t}^\top W_t^2 V_{WW} + \tfrac{1}{2}\nu_\mu^2 \vartheta_t V_{\mu\mu} + \tfrac{1}{2}\nu_\vartheta^2 \vartheta_t V_{\vartheta\vartheta} \\ + \sigma_{R,t}^{(\mu)} W_t \nu_\mu \sqrt{\vartheta_t} V_{W\mu} + \sigma_{R,t}^{(\vartheta)} W_t \nu_\vartheta \sqrt{\vartheta_t} V_{W\vartheta}. \end{aligned} \tag{83}$$

Substituting (80), dividing throughout by V, and using (82) we obtain

$$\begin{aligned} 0 = \frac{1-\gamma}{1-1/\psi} \left(H(\mu_t, \vartheta_t)^{\frac{1-\psi}{1-\gamma}} - \rho \right) + (1-\gamma)(\mu_{R,t} - C_t/W_t) - \tfrac{1}{2}\gamma(1-\gamma) \boldsymbol{\sigma}_{R,t} \boldsymbol{\sigma}_{R,t}^\top \\ +(1-\gamma)\sigma_{R,t}^{(\mu)} \nu_\mu \sqrt{\vartheta_t} \frac{H_\mu(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} + \kappa_\mu (\bar{\mu} - \mu_t) \frac{H_\mu(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} + \tfrac{1}{2}\nu_\mu^2 \vartheta_t \frac{H_{\mu\mu}(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} \\ +(1-\gamma)\sigma_{R,t}^{(\vartheta)} \nu_\vartheta \sqrt{\vartheta_t} \frac{H_\vartheta(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} + \kappa_\vartheta (\bar{\vartheta} - \vartheta_t) \frac{H_\vartheta(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} + \tfrac{1}{2}\nu_\vartheta^2 \vartheta_t \frac{H_{\vartheta\vartheta}(\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)}. \end{aligned} \tag{84}$$

Market clearing implies that $P_t^d = W_t$ (see Proposition A.16). Hence, the price of a claim to consumption is given by

$$P_t^d = C_t H(\mu_t, \vartheta_t)^{-\frac{1-\psi}{1-\gamma}}. \tag{85}$$

with dynamics

$$\frac{dP_t^d}{P_t^d} = \mu_{P,t}dt + \boldsymbol{\sigma}_{P,t}d\mathbf{B}_t, \tag{86}$$

where

$$\begin{aligned}\mu_{P,t} = \mu_t &- \frac{1-\psi}{1-\gamma}\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\kappa_\mu(\bar{\mu}-\mu_t) - \frac{1-\psi}{1-\gamma}\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\kappa_\vartheta(\bar{\vartheta}-\vartheta_t) \\ &- \tfrac{1}{2}\frac{1-\psi}{1-\gamma}\left[\left(-\frac{1-\psi}{1-\gamma}-1\right)\left(\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right)^2 + \frac{H_{\mu\mu}(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right]\nu_\mu^2\vartheta_t \\ &- \tfrac{1}{2}\frac{1-\psi}{1-\gamma}\left[\left(-\frac{1-\psi}{1-\gamma}-1\right)\left(\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right)^2 + \frac{H_{\vartheta\vartheta}(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right]\nu_\vartheta^2\vartheta_t, \end{aligned}\tag{87}$$

and

$$\boldsymbol{\sigma}_{P,t} = \left[\sqrt{\vartheta_t}\,, -\frac{1-\psi}{1-\gamma}\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\nu_\mu\sqrt{\vartheta_t}\,, -\frac{1-\psi}{1-\gamma}\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\nu_\vartheta\sqrt{\vartheta_t}\right]. \tag{88}$$

By combining the consumption to price ratio from (85) and the price process in (86), the equilibrium dynamic of the return for the risky asset in (71) becomes

$$dR_t^d \equiv \frac{dP_t^d}{P_t^d} + \frac{C_t}{P_t^d}dt = \underbrace{(\mu_{P,t} + C_t/W_t)}_{=\mu_{R,t}}dt + \underbrace{\boldsymbol{\sigma}_{P,t}}_{=\boldsymbol{\sigma}_{R,t}}d\mathbf{B}_t. \tag{89}$$

Substituting $\mu_{R,t}$ and $\boldsymbol{\sigma}_{R,t}$ into (84) yields (81) and verifies the form (80). ■

Proposition A.15 (Optimal wealth-consumption ratio) *Given the value function* (80)*, the optimal wealth-consumption ratio is given by*

$$\frac{W_t}{C_t} = H(\mu_t,\vartheta_t)^{-\frac{1-\psi}{1-\gamma}}. \tag{90}$$

Proof. Take the partial derivatives of the normalized aggregator in (9) and of the value function in (80). Substitute into the first order condition (77) to obtain (90). ■

Proposition A.16 (Risky asset) *Consider a claim on dividends, which continuously pays $X_{d,t} = C_t$. In general equilibrium, the price of this claim is*

$$P_t^d = C_t H(\mu_t,\vartheta_t)^{-\frac{1-\psi}{1-\gamma}}. \tag{91}$$

Proof. In equilibrium, the consumer's wealth is defined by the present value of all future consumption which, in the absence of arbitrage, defines the price of an asset that pays consumption as its dividend, P_t^d. Hence, (90) implies that

$$P_t^d = C_t H(\mu_t,\vartheta_t)^{-\frac{1-\psi}{1-\gamma}}. \tag{92}$$

On the other hand, to obtain (92) note that market clearing implies that financial wealth in (73) evolves over time according to

$$
\begin{aligned}
dW_t &= (\mu_{R,t}W_t - C_t)\, dt + \boldsymbol{\sigma}_{R,t} W_t d\mathbf{B}_t \\
\Leftrightarrow \quad \frac{dW_t + C_t dt}{W_t} &= \mu_{R,t} dt + \boldsymbol{\sigma}_{R,t} d\mathbf{B}_t. \qquad (93)
\end{aligned}
$$

By comparing (71) and (93) it follows that $P_t^d = W_t$, and using (90) the result obtains. ■

Proposition A.17 (Risk-free rate) *The instantaneous return on the risk-free asset is*

$$
r_t^f = \mu_{R,t} - \gamma \boldsymbol{\sigma}_{R,t} \boldsymbol{\sigma}_{R,t}^{\top} + \frac{H_\mu (\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} \sigma_{R,t}^{(\mu)} \nu_\mu \sqrt{\vartheta_t} + \frac{H_\vartheta (\mu_t, \vartheta_t)}{H(\mu_t, \vartheta_t)} \sigma_{R,t}^{(\vartheta)} \nu_\vartheta \sqrt{\vartheta_t}. \qquad (94)
$$

Proof. Using the condition for the portfolio share (78) together with the market clearing condition $\theta_t = 1$, and the derivatives of the value function (80) the desired result follows. ■

The function $H(\mu_t, \vartheta_t)$ in (80) only admits a closed form solution for the case of unitary elasticity of intertemporal substitution, $\psi = 1$, or the limiting case of CRRA, $\psi = 1/\gamma$. For any other values of ψ we use a log-linear approximation of the unknown function around the unconditional mean of the state variables (a similar approach is taken by Campbell, Chacko, Rodriguez, and Viceira, 2004; Chacko and Viceira, 2005).

Proposition A.18 (Log-linear approximation) *For $\psi \neq 1$, the equilibrium value function in* (80) *can be approximated by*

$$
V(W_t, \mu_t, \vartheta_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp\left(a_H + b_H \mu_t + c_H \vartheta_t\right), \qquad (95)
$$

where the approximation constants solve the system of equations

$$
\begin{aligned}
a_H &= \frac{1}{\psi h_1} \left(\frac{1-\gamma}{1-1/\psi} h_0 - \rho \frac{1-\gamma}{1-1/\psi} + \psi \kappa_\mu \bar{\mu} b_H + \psi \kappa_\vartheta \bar{\vartheta} c_H \right) \\
b_H &= \frac{1-\gamma}{\psi} \frac{1}{\kappa_\mu + h_1} \\
c_H &= \frac{\psi (h_1 + \kappa_\vartheta)}{\nu_\vartheta^2 \psi^2} \pm \sqrt{\left(\frac{\psi (h_1 + \kappa_\vartheta)}{\nu_\vartheta^2 \psi^2} \right)^2 - \frac{\nu_\mu^2 \psi^2 b_H^2 - \gamma (1-\gamma)}{\nu_\vartheta^2 \psi^2}}.
\end{aligned}
$$

The linearization constants h_0 and h_1 are given by

$$
h_1 = \exp\left(\frac{1-\psi}{1-\gamma} \left(a_H + b_H \bar{\mu} + c_H \bar{\vartheta} \right) \right), \text{ and } h_0 = h_1 (1 - \ln h_1),
$$

where $\bar{\mu} = E(\mu_t)$ and $\bar{\vartheta} = E(\vartheta_t)$.

Proof. Defining $c_t \equiv \ln C_t$ and $w_t \equiv \ln W_t$, the consumption-wealth ratio $C_t/W_t = \exp(c_t - w_t)$ is approximated by a log-linear approximation around the mean consumption-wealth ratio

$$\exp(c_t - w_t) \approx h_1 + h_1(c_t - w_t - \ln h_1) = h_0 + h_1(c_t - w_t).$$

where $h_1 = \exp(\overline{c-w})$, $\overline{c-w}$ is the log consumption-wealth ratio and $h_0 = h_1(1-\ln h_1)$.

Taking logs in (90) we obtain

$$c_t - w_t = \frac{1-\psi}{1-\gamma}\ln H(\mu_t, \vartheta_t),$$

and substituting it out into the approximation yields

$$C_t/W_t \approx h_0 + h_1\frac{1-\psi}{1-\gamma}\ln H(\mu_t, \vartheta_t) = h_0 + h_1\frac{1-\psi}{1-\gamma}\ln H(\mu_t, \vartheta_t).$$

Solving $H(\mu_t, \vartheta_t)$. Consider now the approximation of the function $H(\mu_t, \vartheta_t)$ that solves the maximized Bellman equation in (81), that is,

$$\begin{aligned}
0 \;=\; & \frac{1-\gamma}{1-1/\psi}\overbrace{H(\mu_t,\vartheta_t)^{\frac{1-\psi}{1-\gamma}}}^{=C_t/W_t} - \rho\frac{1-\gamma}{1-1/\psi} + (1-\gamma)\mu_t - \tfrac{1}{2}\gamma(1-\gamma)\vartheta_t \\
& + \psi\kappa_\mu(\bar{\mu}-\mu_t)\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)} + \psi\kappa_\vartheta(\bar{\vartheta}-\vartheta_t)\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)} - \tfrac{1}{2}\psi(1-\psi)\nu_\mu^2\vartheta_t\left(\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right)^2 \\
& - \tfrac{1}{2}\psi(1-\psi)\nu_\vartheta^2\vartheta_t\left(\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right)^2 + \tfrac{1}{2}\psi\nu_\mu^2\vartheta_t\frac{H_{\mu\mu}(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)} + \tfrac{1}{2}\psi\nu_\vartheta^2\vartheta_t\frac{H_{\vartheta\vartheta}(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}.
\end{aligned}$$

Substitute the log-linear approximation to the consumption-wealth ratio to arrive at

$$\begin{aligned}
0 \;=\; & \frac{1-\gamma}{1-1/\psi}h_0 - \psi h_1\ln H(\mu_t,\vartheta_t) - \rho\frac{1-\gamma}{1-1/\psi} + (1-\gamma)\mu_t \\
& - \tfrac{1}{2}\gamma(1-\gamma)\vartheta_t + \psi\kappa_\mu(\bar{\mu}-\mu_t)\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)} + \psi\kappa_\vartheta(\bar{\vartheta}-\vartheta_t)\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)} \\
& - \tfrac{1}{2}\psi(1-\psi)\nu_\mu^2\vartheta_t\left(\frac{H_\mu(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right)^2 - \tfrac{1}{2}\psi(1-\psi)\nu_\vartheta^2\vartheta_t\left(\frac{H_\vartheta(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}\right)^2 \\
& + \tfrac{1}{2}\psi\nu_\mu^2\vartheta_t\frac{H_{\mu\mu}(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)} + \tfrac{1}{2}\psi\nu_\vartheta^2\vartheta_t\frac{H_{\vartheta\vartheta}(\mu_t,\vartheta_t)}{H(\mu_t,\vartheta_t)}.
\end{aligned} \tag{96}$$

We now conjecture that the function $H(\mu_t, \vartheta_t)$ that solves (96) takes the form

$$H(\mu_t, \vartheta_t) = \exp(a_H + b_H\mu_t + c_H\vartheta_t), \tag{97}$$

implying that after we collected terms

$$\begin{aligned}
0 \;=\; & \frac{1-\gamma}{1-1/\psi}h_0 - \psi h_1 a_H - \rho\frac{1-\gamma}{1-1/\psi} + \psi\kappa_\mu\bar{\mu}b_H + \psi\kappa_\vartheta\bar{\vartheta}c_H \\
& + \big((1-\gamma) - \psi(\kappa_\mu + h_1)b_H\big)\mu_t \\
& + \big(\tfrac{1}{2}\nu_\vartheta^2\psi^2 c_H^2 - \psi(h_1+\kappa_\vartheta)c_H + \tfrac{1}{2}\nu_\mu^2\psi^2 b_H^2 - \tfrac{1}{2}\gamma(1-\gamma)\big)\vartheta_t.
\end{aligned}$$

Using the method of undetermined coefficients, the solution for b_H (making the coefficient on μ_t zero) is given by

$$b_H = \frac{1-\gamma}{\psi}\frac{1}{\kappa_\mu + h_1}. \tag{98}$$

Given b_H, the solution for c_H (making the coefficient on ϑ_t zero) is given by

$$c_H = \frac{\psi\left(h_1 + \kappa_\vartheta\right)}{\nu_\vartheta^2 \psi^2} \pm \sqrt{\left(\frac{\psi\left(h_1 + \kappa_\vartheta\right)}{\nu_\vartheta^2 \psi^2}\right)^2 - \frac{\nu_\mu^2 \psi^2 b_H^2 - \gamma\left(1-\gamma\right)}{\nu_\vartheta^2 \psi^2}}, \tag{99}$$

and that for a_H (making the constant term zero) is

$$a_H = \frac{1}{\psi h_1}\left(\frac{1-\gamma}{1-1/\psi} h_0 - \rho \frac{1-\gamma}{1-1/\psi} + \psi \kappa_\mu \bar{\mu} b_H + \psi \kappa_\vartheta \bar{\vartheta} c_H\right). \tag{100}$$

The values of a_H, b_H and c_H that solve the PDE in (96) depend on the optimal expected log consumption-wealth ratio $\ln h_1$, which is endogenous to the model. Given the conjecture for $H\left(\mu_t, \vartheta_t\right)$, we have that for $\psi \neq 1$

$$\begin{aligned} h_1 &= \exp\left(E\left(c_t - w_t\right)\right) \\ &= \exp\left(E\left(\frac{1-\psi}{1-\gamma}\left(a_H + b_H \mu_t + c_H \vartheta_t\right)\right)\right) \\ &= \exp\left(\frac{1-\psi}{1-\gamma}\left(a_H + b_H \bar{\mu} + c_H \bar{\vartheta}\right)\right), \end{aligned}$$

where $\bar{\mu} = E\left(\mu_t\right)$ and $\bar{\vartheta} = E\left(\vartheta_t\right)$. ■

A.5 Tables and Figures

Table A.1: C-CAPM Estimates (Two-assets case, R_t^b, R_t^c)

The table reports efficient GMM estimates and the Euler equation errors (RMSE, annualized). For the conditional moments and restrictions we use $q = 0$, $\bar{\lambda} = 0$, $\bar{\nu} = -0.4$, $\alpha = 0.33$, $\nu = -0.75$ and $\lambda = 0.017$. For the conditional GMM (Production) the free parameter δ is used to satisfy the theoretical restriction $\rho = \bar{\rho}$. For the Euler equation (EE) errors, we fix $\rho = 0.03$ and $\gamma = 4$. For the Parker-Julliard approach we fix $\rho = 0.03$ and $\alpha_1 = -0.01$ ($\alpha_1 = -0.05$ annual data). Asymptotic t-values are below the estimates (RMSE/RMSR in brackets). GMM iterates until convergence.

Parameter Estimates from Empirical Data																
		GMM			conditional GMM						EE errors			Parker/Julliard		
		Model-free			Endowment			Production			Model-free			Model-free		
Data		β	γ	RMSE	β	γ	RMSE	β	γ	RMSE	α_1	α_2	RMSE	α_0	γ	RMSE
US	1951:Q4-2016:Q4	1.093	61.7	0.0	0.985	4.1	0.0	0.995	4.0	0.0	−0.009	−0.002	0.0	0.007	4.2	0.0
		2.0	0.1	[0.00]	6.7	0.1	[0.00]	29.8	0.1	[0.00]	−0.3	−0.0	[0.00]	0.1	0.4	[0.00]
US	1951:Q4-2002:Q4	1.094	47.2	0.0	0.988	2.6	0.0	0.992	2.7	0.0	−0.008	−0.004	0.0	0.004	4.5	0.0
		0.9	0.0	[0.00]	14.4	0.1	[0.00]	16.2	0.1	[0.00]	−0.3	−0.1	[0.00]	0.0	0.4	[0.00]
US	1951:Q4-2000:Q4	1.069	91.1	1.1	0.986	3.7	0.0	0.993	3.6	0.0	−0.008	−0.002	0.0	0.006	4.5	0.0
		0.0	0.0	[0.16]	8.0	0.1	[0.00]	23.2	0.1	[0.00]	−0.3	−0.0	[0.00]	0.1	0.4	[0.00]
US	(a) 1900-2008	0.800	34.1	0.0	0.890	5.9	0.0	0.991	6.1	0.0	−0.053	0.007	0.0	0.064	3.8	0.0
		0.3	0.3	[0.00]	2.0	0.7	[0.00]	4.7	0.6	[0.00]	−0.4	0.0	[0.00]	0.3	0.4	[0.00]
CAN	(a) 1934-2008	1.387	49.6	4.8	0.916	6.0	0.0	1.027	6.2	0.0	−0.070	−0.010	0.0	0.065	3.0	0.0
		0.0	0.0	[0.53]	2.7	1.0	[0.00]	6.0	0.8	[0.00]	−0.7	−0.1	[0.00]	0.4	0.7	[0.00]
GER	(a) 1900-2008	0.652	6.6	6.7	0.702	3.7	4.3	0.728	3.4	3.8	0.408	0.367	0.0	−0.152	−2.1	26.7
		0.0	0.0	[0.28]	0.0	0.0	[0.18]	0.0	0.0	[0.16]	0.1	0.1	[0.00]	−0.0	−0.0	[1.09]
ITA	(a) 1900-2008	1.002	3.9	8.3	0.753	7.5	0.0	0.934	8.3	0.0	−0.032	0.080	0.0	0.116	5.5	0.0
		0.0	0.0	[0.57]	0.5	0.5	[0.00]	1.4	0.4	[0.00]	−0.2	0.1	[0.00]	0.2	0.3	[0.00]
JAP	(a) 1900-2008	1.011	3.4	9.3	0.576	7.8	0.0	0.613	9.0	0.0	−0.019	0.090	0.0	0.108	3.1	3.1
		0.0	0.0	[0.75]	0.2	0.8	[0.00]	0.1	0.5	[0.00]	−0.0	0.1	[0.00]	0.0	0.0	[0.25]
UK	(a) 1900-2008	0.579	37.5	0.0	0.926	4.4	0.0	0.972	4.3	0.0	−0.030	0.002	0.0	0.031	6.1	0.0
		0.0	0.1	[0.00]	1.7	0.2	[0.00]	7.2	0.2	[0.00]	−0.2	0.0	[0.00]	0.1	0.3	[0.00]
US	(a) 1951-2008	1.143	8.7	4.5	0.910	6.1	0.0	1.023	6.3	0.0	−0.066	−0.005	0.0	0.066	3.2	0.0
		0.0	0.0	[0.49]	2.5	0.9	[0.00]	8.8	0.7	[0.00]	−0.9	−0.0	[0.00]	0.4	1.1	[0.00]
CAN	(a) 1951-2008	0.828	90.6	0.0	0.917	5.5	0.0	0.999	5.5	0.0	−0.049	−0.000	0.0	0.051	4.1	0.0
		0.1	0.2	[0.00]	2.9	0.7	[0.00]	7.7	0.6	[0.00]	−0.5	−0.0	[0.00]	0.3	0.8	[0.00]
GER	(a) 1951-2008	1.867	69.8	0.0	0.877	7.1	0.0	1.067	7.7	0.0	−0.093	0.004	0.0	0.111	2.3	0.0
		0.3	0.2	[0.00]	1.6	1.1	[0.00]	5.1	0.8	[0.00]	−0.9	0.0	[0.00]	0.4	1.0	[0.00]
ITA	(a) 1951-2008	1.541	24.0	0.0	0.948	5.2	0.0	1.021	5.2	0.0	−0.062	−0.017	0.0	0.049	3.5	0.0
		0.5	0.2	[0.00]	2.7	0.4	[0.00]	3.9	0.3	[0.00]	−0.6	−0.1	[0.00]	0.2	0.9	[0.00]
JAP	(a) 1951-2008	2.552	39.9	0.0	0.981	6.8	0.0	1.163	7.2	0.0	−0.137	−0.053	0.0	0.107	1.6	0.0
		0.4	0.2	[0.00]	2.2	1.0	[0.00]	3.7	0.8	[0.00]	−1.3	−0.2	[0.00]	0.4	1.5	[0.00]
UK	(a) 1951-2008	1.423	28.5	0.0	0.906	5.5	0.0	0.991	5.6	0.0	−0.040	0.013	0.0	0.055	4.5	0.0
		0.5	0.2	[0.00]	1.8	0.4	[0.00]	7.3	0.3	[0.00]	−0.5	0.0	[0.00]	0.2	0.8	[0.00]

Table A.2: C-CAPM Estimates (Larger cross-section)

The table reports GMM estimates and the Euler equation errors (RMSE, annualized) with identity weighting matrix. For the Euler equation (EE) errors, we use two assets and add 6 Fama-French portfolios sorted by size and book-to-market (Big Value, Big Neutral, Big Growth, Small Value, Small Neutral, Small Growth) to either estimate β, γ, and $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_8$ or $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ by fixing either $\rho = 0.03$ and $\gamma = 4$, or 3 out of the 6 FF portfolios (Big Value, Big Neutral, Small Value) to underpredict their returns by the same amount $\alpha_3 \equiv \alpha_4 \equiv \alpha_6$. Asymptotic t-values are below the estimates (RMSE/RMSR in brackets).

Parameter Estimates from Empirical Data											
US Data: 1951:Q4-2016:Q4											
Model	β	γ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	RMSE
GMM	0.846	147.1									1.7
											[0.13]
EE errors			−0.009	−0.002	0.007	0.009	0.013	0.009	0.017	0.021	0.0
			−0.3	−0.0	0.1	0.1	0.1	0.1	0.2	0.2	[0.00]
EE errors	0.989	4.6	−0.015	−0.008	0.002		0.007		0.011	0.015	0.3
	11.7	60.6	−0.4	−0.1	0.0		0.1		0.1	0.1	[0.02]

Table A.3: Simulation study (rare events model - endowment economy)

		(1)	(2)
ρ	rate of time preference	0.03	0.03
γ	coef. of relative risk aversion	4	4
$\bar{\mu}$	consumption growth	0.01	0.01
$\bar{\sigma}$	consumption noise	0.005	0.005
$-\bar{\nu}$	size of consumption disaster	0.4	0
λ	consumption disaster probability	0.017	0
$-\kappa$	size of government default	0.3	0
q	default probability	0.5	0

Table A.4: Simulation study (rare events model - production economy)

		(1)	(2)
ρ	rate of time preference	0.024	0.016
γ	coef. of relative risk aversion	4	4
α	output elasticity of capital	0.6	0.6
δ	capital depreciation	0.025	0.025
$\bar{\mu}$	productivity growth	0.01	0.01
$\bar{\sigma}$	productivity noise	0.01	0.01
$-\bar{\nu}$	size of productivity slump	0.01	0
$\bar{\lambda}$	productivity jump probability	0.2	0
σ	capital stochastic depreciation	0.005	0.005
$-\nu$	size of capital disaster	0.55	0.55
λ	capital disaster probability	0.017	0.017

Table A.5: Simulation study (long-run risk model)

		(1)	(2)
ρ	rate of time preference	0.024	0.03
γ	coef. of relative risk aversion	10	10
ψ	EIS	1.5	1.5
$\bar{\mu}$	consumption growth	0.018	0.018
κ_μ	LRR persistence	0.256	0.3
ν_μ	LRR volatility multiple	0.528	0.456
$\bar{\vartheta}$	baseline volatility ($\times 100$)	0.0729	0.0625
κ_ϑ	persistence volatility	0.156	0.015
ν_ϑ	vol-of-vol	0.0035	0.0027

Table A.6: C-CAPM simulation results (rare events - endowment economy)

The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM (conditional on no disasters) observed at quarterly frequency in the endowment economy with rare events (cf. Section 3.1) for a parameterization as in column (1) in Table A.3; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.97$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5,000 Monte Carlo sample paths, each of length 50 years.

Results	analytical solution	conditional (no disasters)			
	parameterization (1)	Mean	Std. dev.	Mode	Median
e_R^b	EE error risky bond	−5.59	0.28	−5.48	−5.59
e_X^c	EE error excess return	1.66	0.07	1.66	1.66
$RMSE$	root mean square error	4.12	0.20	4.04	4.12
Observed random variables					
R_{t+1}^b	bill return	1.35	0.00	1.50	1.35
R_{t+1}^c	equity return	3.05	0.07	3.05	3.05
$R_{t+1}^c - R_{t+1}^b$	equity premium	1.70	0.07	1.70	1.70
$\ln(C_{t+1}/C_t)$	consumption growth	1.00	0.07	1.00	1.00
Parameter estimates					
$\hat{\beta}$	factor of time preference	1.17	0.17	1.21	1.19
$\hat{\gamma}$	coef. of relative risk aversion	804.64	244.70	752.50	754.20
$\widehat{e_R^b}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e_X^c}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{RMSE}$	root mean square error	0.00	0.00	0.00	0.00

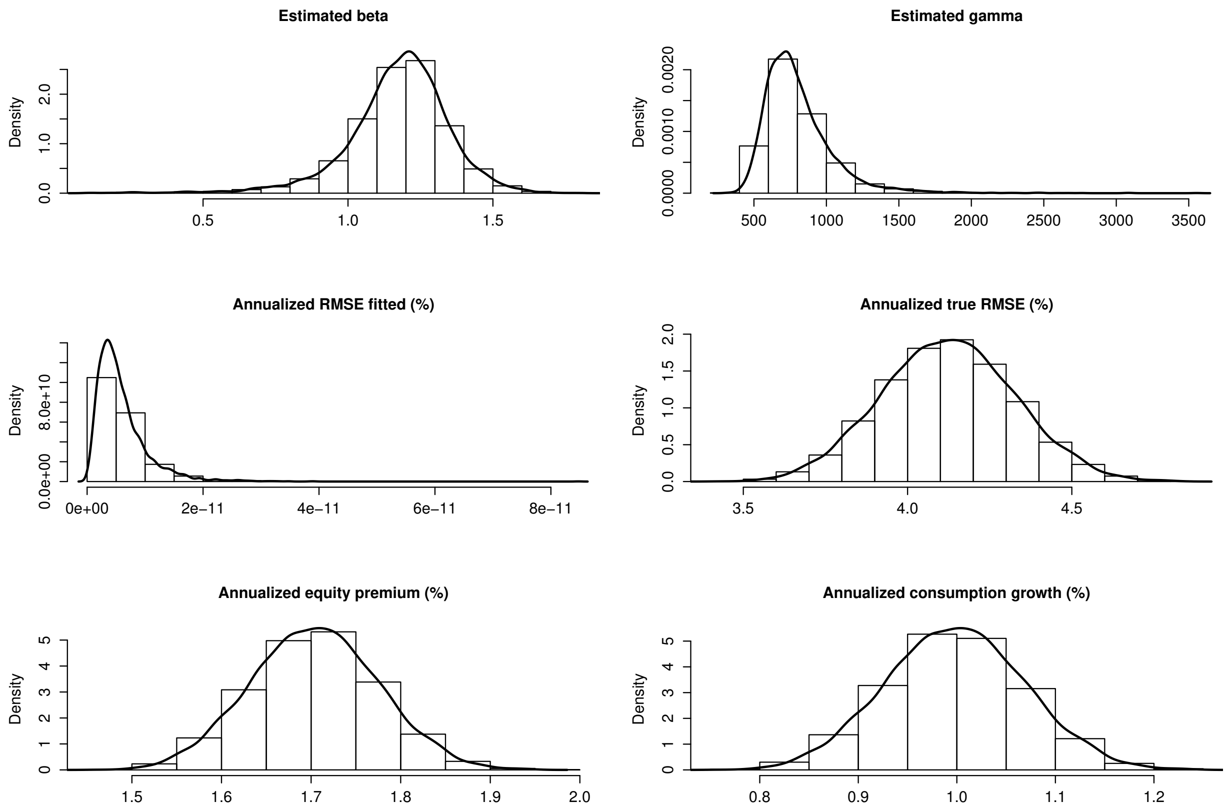

Table A.7: C-CAPM simulation results (rare events - endowment economy)

The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM (conditional on no disasters) observed at quarterly frequency in the endowment economy with rare events (cf. Section 3.1) for a parameterization as in column (2) in Table A.3; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.97$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5,000 Monte Carlo sample paths, each of length 50 years.

Results	analytical solution	conditional (no disasters)			
	parameterization (2)	Mean	Std. dev.	Mode	Median
e^b_R	EE error risky bond	0.00	0.29	0.06	0.00
e^c_X	EE error excess return	0.00	0.07	−0.02	0.00
$RMSE$	root mean square error	0.17	0.13	0.09	0.14
Observed random variables					
R^b_{t+1}	bill return	7.04	0.00	7.50	7.04
R^c_{t+1}	equity return	7.05	0.07	7.06	7.05
$R^c_{t+1} - R^b_{t+1}$	equity premium	0.01	0.07	0.01	0.01
$\ln(C_{t+1}/C_t)$	consumption growth	1.00	0.07	0.98	1.00
Parameter estimates					
$\hat{\beta}$	factor of time preference	1.00	0.07	0.98	0.99
$\hat{\gamma}$	coef. of relative risk aversion	3.73	29.37	9.25	3.64
$\widehat{e^b_R}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e^c_X}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{RMSE}$	root mean square error	0.00	0.00	0.00	0.00

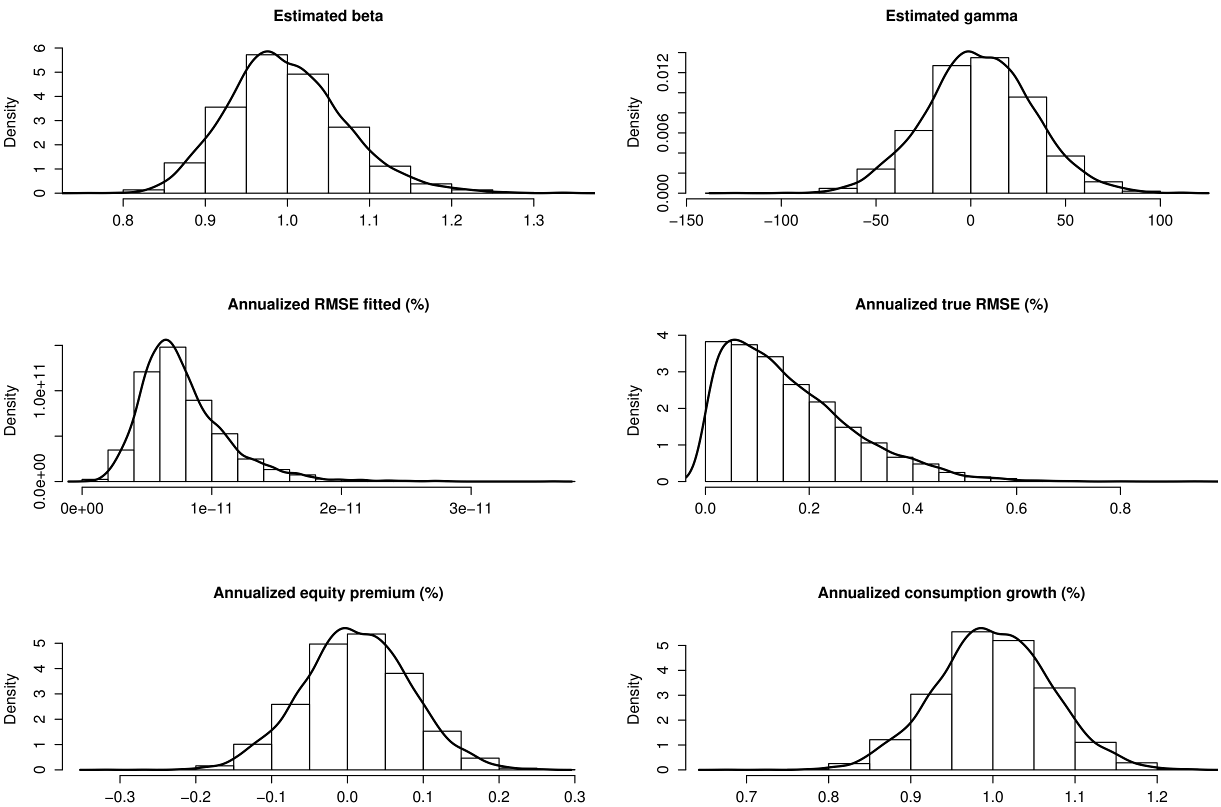

Table A.8: C-CAPM simulation results (rare events - production economy)

The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM (conditional on no disasters) observed at quarterly frequency in the production economy with rare events (cf. Section 3.2) for a parameterization as in column (1) in Table A.4; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5,000 Monte Carlo sample paths, each of length 50 years.

Results	constant-saving-function, parameterization (1)	conditional (no disasters) Mean	Std. dev.	Mode	Median
e_R^b	EE error risky bond	-4.62	0.65	-4.68	-4.61
e_X^c	EE error excess return	4.64	0.17	4.68	4.64
$RMSE$	root mean square error	4.64	0.35	4.62	4.63
Observed random variables					
R_{t+1}^b	bill return (gross)	6.42	0.39	6.43	6.42
R_{t+1}^c	equity return (gross)	11.21	0.40	11.35	11.20
$R_{t+1}^c - R_{t+1}^b$	equity premium	4.79	0.17	4.77	4.78
$\ln(C_{t+1}/C_t)$	consumption growth	2.19	0.24	2.09	2.19
Parameter estimates					
$\hat{\beta}$	factor of time preference	0.80	0.64	0.00	0.86
$\hat{\gamma}$	coef. of relative risk aversion	474.36	442.44	175.00	327.02
$\widehat{e_R^b}$	EE error risky bond	-0.03	0.02	0.00	-0.04
$\widehat{e_X^c}$	EE excess return	3.42	1.31	0.00	3.90
$\widehat{RMSE}$	root mean square error	2.42	0.93	0.00	2.76

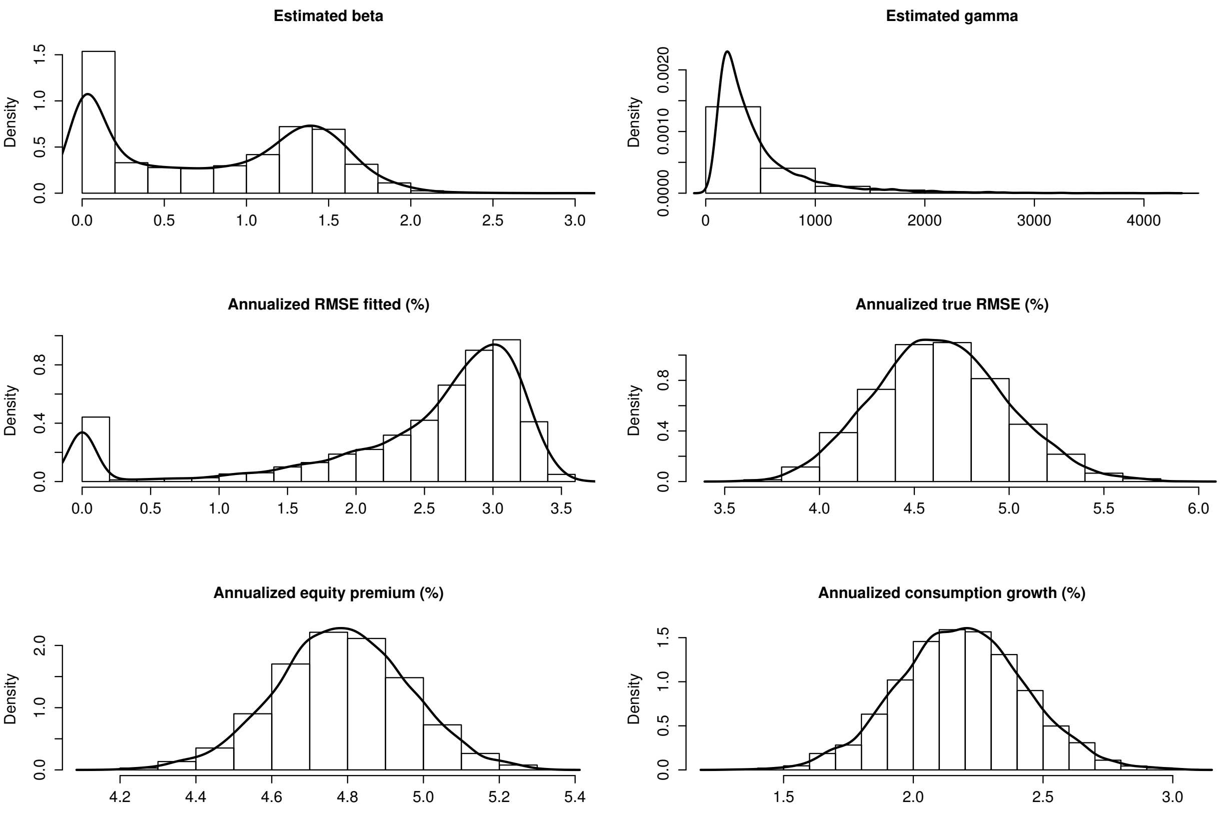

Table A.9: C-CAPM simulation results (rare events - production economy)

The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM (conditional on no disasters) observed at quarterly frequency in the production economy with rare events (cf. Section 3.2) for a parameterization as in column (2) in Table A.4; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5,000 Monte Carlo sample paths, each of length 50 years.

Results	constant-saving-function, parameterization (2)	conditional (no disasters) Mean	Std. dev.	Mode	Median
e_R^b	EE error risky bond	−4.49	0.58	−4.39	−4.49
e_X^c	EE error excess return	4.63	0.17	4.69	4.63
$RMSE$	root mean square error	4.57	0.32	4.47	4.56
Observed random variables					
R_{t+1}^b	bill return (gross)	6.86	0.35	6.81	6.86
R_{t+1}^c	equity return (gross)	11.63	0.37	11.88	11.63
$R_{t+1}^c - R_{t+1}^b$	equity premium	4.78	0.18	4.83	4.78
$\ln(C_{t+1}/C_t)$	consumption growth	2.50	0.22	2.55	2.50
Parameter estimates					
$\hat{\beta}$	factor of time preference	0.88	0.87	0.00	0.57
$\hat{\gamma}$	coef. of relative risk aversion	806.11	661.98	325.00	590.05
$\widehat{e_R^b}$	EE error risky bond	−0.03	0.02	0.00	−0.03
$\widehat{e_X^c}$	EE excess return	2.82	1.45	0.00	3.31
$\widehat{RMSE}$	root mean square error	1.99	1.03	0.00	2.34

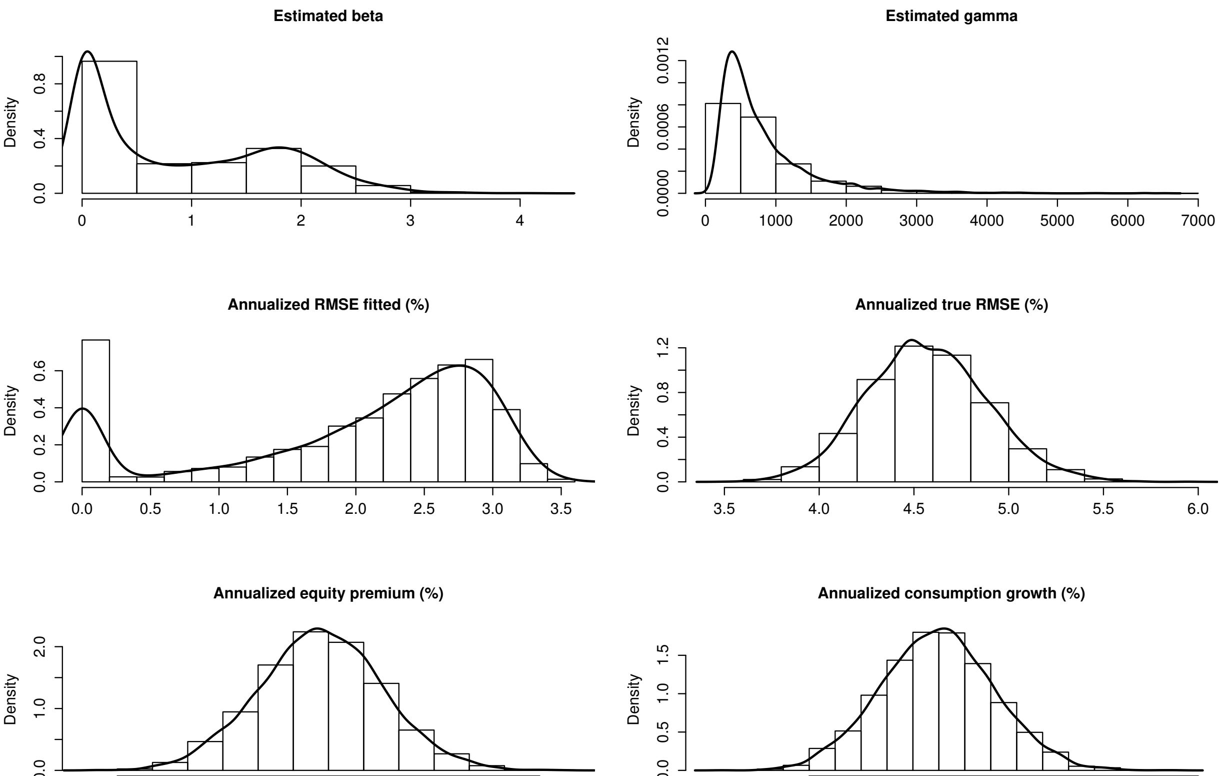

Table A.10: C-CAPM simulation results (long-run risk model)

The table reports the simulated Euler equation (EE) errors and RMSE* (both annualized) for the standard C-CAPM observed at quarterly frequency in the endowment economy with long-run risk (cf. Appendix A.4) for a parameterization as in column (1) in Table A.5; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 10$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5,000 Monte Carlo sample paths, each of length 50 years.

Results	approximate solution parameterization (1)	unconditional Mean	Std. dev.	Mode	Median
$R^b_{t+1} - E(R^b_{t+1})$	pricing error bond	0.00	0.51	0.12	0.00
$R^d_{t+1} - E(R^d_{t+1})$	pricing error risky asset	0.00	0.84	−0.09	−0.01
$RMSE^*$	root mean square error	0.61	0.43	0.25	0.51
Observed random variables					
R^b_{t+1}	bill return	2.58	0.51	2.67	2.58
R^d_{t+1}	equity return	4.17	0.84	4.40	4.16
$R^d_{t+1} - R^b_{t+1}$	equity premium	1.59	0.48	1.50	1.59
$\ln(C_{t+1}/C_t)$	consumption growth	1.76	0.85	1.65	1.76
Parameter estimates					
$\hat{\beta}$	factor of time preference	1.05	0.05	1.03	1.04
$\hat{\gamma}$	coef. of relative risk aversion	21.81	6.95	19.35	21.46
$\widehat{e^b_R}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e^c_X}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{RMSE}$	root mean square error	0.00	0.00	0.00	0.00

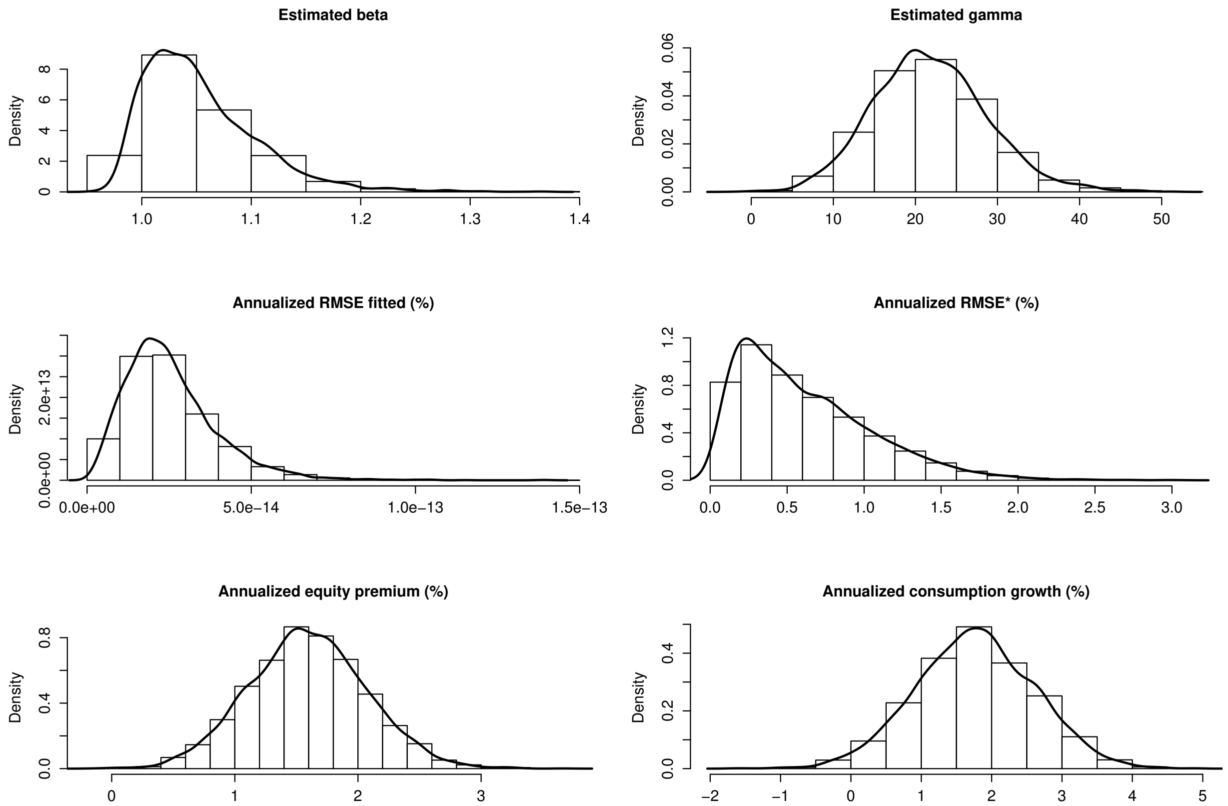

Table A.11: C-CAPM simulation results (long-run risk model)

The table reports the simulated Euler equation (EE) errors and RMSE* (both annualized) for the standard C-CAPM observed at quarterly frequency in the endowment economy with long-run risk (cf. Appendix A.4) for a parameterization as in column (2) in Table A.5; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.97$ and $\gamma = 10$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5,000 Monte Carlo sample paths, each of length 50 years.

Results	approximate solution parameterization (2)	unconditional Mean	Std. dev.	Mode	Median
$R^b_{t+1} - E(R^b_{t+1})$	pricing error bond	0.00	0.46	−0.02	0.05
$R^d_{t+1} - E(R^d_{t+1})$	pricing error risky asset	0.00	0.64	0.12	−0.01
$RMSE^*$	root mean square error	0.49	0.32	0.24	0.42
Observed random variables					
R^b_{t+1}	bill return	3.33	0.46	3.53	3.37
R^d_{t+1}	equity return	4.71	0.64	4.74	4.70
$R^d_{t+1} - R^b_{t+1}$	equity premium	1.38	0.52	1.10	1.33
$\ln(C_{t+1}/C_t)$	consumption growth	1.77	0.63	2.05	1.77
Parameter estimates					
$\hat{\beta}$	factor of time preference	1.06	0.05	1.04	1.05
$\hat{\gamma}$	coef. of relative risk aversion	22.89	7.96	21.65	22.24
$\widehat{e^b_R}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e^c_X}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{RMSE}$	root mean square error	0.00	0.00	0.00	0.00