

Technical Appendix

In this appendix we extensively use a special case of the Fortuin, Kasteleyn and Ginibre (1971) inequality¹ (henceforth FKG) in Harris (1960). This inequality (adapted to our setting) states that for any probability measure on R , and increasing

$$\text{functions } f(x) \text{ and } g(x), \quad \int_R f(x)g(x) d\mu(x) \geq \int_R f(x) d\mu(x) \int_R g(x) d\mu(x).$$

Proof of Proposition 2.1

a. Property II \Rightarrow Property I

$$\text{var}(\tilde{x}_t + \tilde{x}_{t+1}) = \text{var}(\tilde{x}_t) + \text{var}(\tilde{x}_{t+1}) + 2\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) < 2\text{var}(\tilde{x}_t) \quad (\text{by Property II})$$

$$\Rightarrow \text{var}(\tilde{x}_{t+1}) + 2\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) < \text{var}(\tilde{x}_t).$$

Since $\text{var}(\tilde{x}_{t+1}) = \text{var}(\tilde{x}_t)$, this implies that $\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) < 0$,

Hence Property I holds.

b. Property I together with the covariance condition (3) implies Property II.

The proof is by induction.

$$\text{Let } j = 1, \quad \text{var}(\tilde{x}_t + \tilde{x}_{t+1}) = \text{var}(\tilde{x}_t) + \text{var}(\tilde{x}_{t+1}) + 2\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) < 2\text{var}(\tilde{x}_t)$$

(this follows by Property I and the fact that $\text{var}(\tilde{x}_t) = \text{var}(\tilde{x}_{t+1})$)

$$\text{Let } j = 2, \quad \text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2})$$

$$= \sum_{j=0}^2 \text{var}(\tilde{x}_{t+j}) + 2\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) + 2\text{cov}(\tilde{x}_t, \tilde{x}_{t+2}) + 2\text{cov}(\tilde{x}_{t+1}, \tilde{x}_{t+2})$$

¹ We thank Awi Federgruen for bringing the FKG inequality to our attention.

By Property I, $2\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) < 0$

and by condition (3), $2\text{cov}(\tilde{x}_t, \tilde{x}_{t+1}) + 2\text{cov}(\tilde{x}_t, \tilde{x}_{t+2}) < 0$

Again by Property I, $2\text{cov}(\tilde{x}_{t+1}, \tilde{x}_{t+2}) < 0$. Therefore,

$$\text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2}) < \sum_{j=0}^2 \text{var}(\tilde{x}_{t+j}) = 3\text{var}(\tilde{x}_t).$$

Suppose, by Property I and condition (3), $\text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \dots + \tilde{x}_{t+j-1}) < j\text{var}(\tilde{x}_t)$

holds. We show by induction that this implies

$$\text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \dots + \tilde{x}_{t+j}) < (j+1)\text{var}(\tilde{x}_t).$$

$$\text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2} + \dots + \tilde{x}_{t+j})$$

$$= \text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \dots + \tilde{x}_{t+j-1}) + \text{var}(\tilde{x}_{t+j}) + 2\sum_{s=0}^{j-2} \text{cov}(\tilde{x}_{t+s}, \tilde{x}_{t+j}) + 2\text{cov}(\tilde{x}_{t+j-1}, \tilde{x}_{t+j})$$

$$< (j)\text{var}(\tilde{x}_t) + \text{var}(\tilde{x}_{t+j}) + 2\sum_{s=0}^{j-2} \text{cov}(\tilde{x}_{t+s}, \tilde{x}_{t+j}) + 2\text{cov}(\tilde{x}_{t+j-1}, \tilde{x}_{t+j})$$

(by induction)

By Property I, $\text{cov}(\tilde{x}_{t+j-1}, \tilde{x}_{t+j}) < 0$. Thus by condition (3),

$$+ 2\sum_{s=0}^{j-2} \text{cov}(\tilde{x}_{t+s}, \tilde{x}_{t+j}) + 2\text{cov}(\tilde{x}_{t+j-1}, \tilde{x}_{t+j}) < 0.$$

Therefore, since $\text{var}(\tilde{x}_{t+j}) = \text{var}(\tilde{x}_t)$

$$\text{var}(\tilde{x}_t + \tilde{x}_{t+1} + \dots + \tilde{x}_{t+j}) < (j+1)\text{var}(\tilde{x}_t).$$

For $\text{I} \Rightarrow \text{II}$, the persistence of the series must rapidly decline.

Proof of Proposition 3.1

- a. The equity price relationship follows from an application of Jensen's inequality:

$$\begin{aligned}
\text{cov}(\tilde{p}_{t-1}^e, \tilde{p}_t^e) &= \text{cov}(\tilde{k}_t, \tilde{k}_{t+1}) = \text{cov}(\tilde{k}_t, \alpha\beta\tilde{k}_t^\alpha\tilde{\lambda}_t) \\
&= \alpha\beta E(\tilde{\lambda}_t) \left\{ E(\tilde{k}_t^{1+\alpha}) - E(\tilde{k}_t) E(\tilde{k}_t^\alpha) \right\} \\
&> \alpha\beta E(\tilde{\lambda}_t) \left\{ E(\tilde{k}_t^{1+\alpha}) - E(\tilde{k}_t) \left(E(\tilde{k}_t^\alpha) \right)^\alpha \right\}, \text{ since } k_t^\alpha \text{ is concave and } E(\tilde{k}_t^\alpha) < \left(E(\tilde{k}_t) \right)^\alpha \\
&\quad \text{by Jensen's inequality.} \\
&= \alpha\beta E(\tilde{\lambda}_t) \left\{ E(\tilde{k}_t^{1+\alpha}) - \left(E(\tilde{k}_t) \right)^{1+\alpha} \right\} > 0.
\end{aligned}$$

- b. Mean aversion in dividends

$$\begin{aligned}
\text{By (17), } \text{cov}(\tilde{d}_t, \tilde{d}_{t+1}) &= \text{cov}(\alpha(1-\beta)\tilde{k}_t^\alpha\tilde{\lambda}_t, \alpha(1-\beta)\tilde{k}_{t+1}^\alpha\tilde{\lambda}_{t+1}) \\
&= \left(\alpha(1-\beta) \right)^2 \text{cov} \left(\tilde{k}_t^\alpha\tilde{\lambda}_t, \left(\alpha\beta\tilde{k}_t^\alpha\tilde{\lambda}_t \right)^\alpha \tilde{\lambda}_{t+1} \right) \\
&= \left(\alpha(1-\beta) \right)^2 \left(\alpha\beta \right)^\alpha \text{cov} \left(\tilde{k}_t^\alpha\tilde{\lambda}_t, \tilde{k}_t^{\alpha^2}\tilde{\lambda}_t^\alpha\tilde{\lambda}_{t+1} \right) \\
&= \left(\alpha(1-\beta) \right)^2 \left(\alpha\beta \right)^\alpha \left\{ E \left(\tilde{k}_t^{\alpha+\alpha^2}\tilde{\lambda}_t^{1+\alpha}\tilde{\lambda}_{t+1} \right) \right. \\
&\quad \left. - E \left(\tilde{k}_t^\alpha\tilde{\lambda}_t \right) E \left(\tilde{k}_t^{\alpha^2}\tilde{\lambda}_t^\alpha\tilde{\lambda}_{t+1} \right) \right\} \\
&= \left(\alpha(1-\beta) \right)^2 \left(\alpha\beta \right)^\alpha E(\tilde{\lambda}_{t+1}) \left\{ E \left(\tilde{k}_t^{\alpha+\alpha^2} \right) E \left(\tilde{\lambda}_t^{1+\alpha} \right) - E \left(\tilde{k}_t^\alpha \right) E \left(\tilde{\lambda}_t \right) E \left(\tilde{k}_t^{\alpha^2} \right) E \left(\tilde{\lambda}_t^\alpha \right) \right\}.
\end{aligned}$$

By FKG or the Harris inequality

$$\begin{aligned}
E \left(\tilde{k}_t^{\alpha+\alpha^2} \right) &\geq E \left(\tilde{k}_t^\alpha \right) E \left(\tilde{k}_t^{\alpha^2} \right), \text{ and} \\
E \left(\tilde{\lambda}_t^{1+\alpha} \right) &\geq E \left(\tilde{\lambda}_t \right) E \left(\tilde{\lambda}_t^\alpha \right). \text{ Thus,}
\end{aligned}$$

$$\left(\alpha(1-\beta)\right)^2 (\alpha\beta)^\alpha E(\tilde{\lambda}_{t+1}) \left\{ E(\tilde{k}_t^{\alpha+\alpha^2}) E(\tilde{\lambda}_t^{1+\alpha}) - E(\tilde{k}_t^\alpha) E(\tilde{\lambda}_t) E(\tilde{k}_t^{\alpha^2}) E(\tilde{\lambda}_t^\alpha) \right\} \geq 0.$$

c. Derivation of risk free bond price.

$$\begin{aligned} p_t^b &= \beta \int \frac{u_1(\tilde{c}_{t+1})}{u_1(c_t)} dF(\tilde{c}_{t+1}, c_t) \\ &= \beta \int \frac{(1-\alpha\beta)k_t^\alpha \lambda_t}{(1-\alpha\beta)[\alpha\beta k_t^\alpha \lambda_t]^\alpha \tilde{\lambda}_{t+1}} dF(\tilde{\lambda}_{t+1}) \\ &= \beta \int \frac{1}{(\alpha\beta)^\alpha [k_t^\alpha \lambda_t]^{\alpha-1} \tilde{\lambda}_{t+1}} dF(\tilde{\lambda}_{t+1}) \\ &= \frac{\beta}{(\alpha\beta)^\alpha} k_t^{\alpha(1-\alpha)} \lambda_t^{1-\alpha} E(\lambda_{t+1}^{-1}), \text{ since } \{\tilde{\lambda}_t\} \text{ is i.i.d.} \end{aligned}$$

Thus, $p_t^b = \frac{\beta E(\tilde{\lambda}_{t+1}^{-1})}{(\alpha\beta)^\alpha} k_t^{\alpha(1-\alpha)} \lambda_t^{1-\alpha}$, where $E(\tilde{\lambda}_{t+1}^{-1})$ is constant for all t . As a result,

we henceforth omit the time subscript from this term.

$$\begin{aligned} \text{cov}(\tilde{p}_t^b, \tilde{p}_{t+1}^b) &= \text{cov}\left(\left[\frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha}\right] \tilde{k}_t^{\alpha(1-\alpha)} \tilde{\lambda}_t^{1-\alpha}, \frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha} \tilde{k}_{t+1}^{\alpha(1-\alpha)} \tilde{\lambda}_{t+1}^{1-\alpha}\right) \\ &= \left[\frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha}\right]^2 \text{cov}\left(\tilde{k}_t^{\alpha-\alpha^2} \tilde{\lambda}_t^{1-\alpha}, \tilde{k}_{t+1}^{\alpha(1-\alpha)} \tilde{\lambda}_{t+1}^{1-\alpha}\right) \\ &= \left[\frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha}\right]^2 \text{cov}\left(\tilde{k}_t^{\alpha-\alpha^2} \tilde{\lambda}_t^{1-\alpha}, (\alpha\beta)^{\alpha-\alpha^2} \tilde{k}_t^{\alpha^2-\alpha^3} \tilde{\lambda}_t^{\alpha-\alpha^2} \tilde{\lambda}_{t+1}^{1-\alpha}\right), \text{ since } k_{t+1} = \alpha\beta k_t^\alpha \lambda_t. \end{aligned}$$

$$= \left[\frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha} \right]^2 (\alpha\beta)^{\alpha-\alpha^2} \left\{ E\left(\tilde{k}_t^{\alpha-\alpha^3} \tilde{\lambda}_t^{1-\alpha^2} \tilde{\lambda}_{t+1}^{1-\alpha}\right) - E\left(\tilde{k}_t^{\alpha-\alpha^2} \tilde{\lambda}_t^{1-\alpha}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha^3} \tilde{\lambda}_t^{\alpha-\alpha^2} \tilde{\lambda}_{t+1}^{1-\alpha}\right) \right\}.$$

Since $\{\lambda_t\}$ is i.i.d. and the fact that k_t is determined in period $t-1$ independent of λ_t or λ_{t+1} , we may equivalently write

$$\begin{aligned} \text{cov}(\tilde{p}_t^b, \tilde{p}_{t+1}^b) &= \left[\frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha} \right]^2 (\alpha\beta)^{\alpha-\alpha^2} \left\{ E\left(\tilde{k}_t^{\alpha-\alpha^3}\right) E\left(\tilde{\lambda}_t^{1-\alpha^2}\right) E\left(\tilde{\lambda}_{t+1}^{1-\alpha}\right) \right. \\ &\quad \left. - E\left(\tilde{k}_t^{\alpha-\alpha^2}\right) E\left(\tilde{\lambda}_t^{1-\alpha}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha^3}\right) E\left(\tilde{\lambda}_t^{\alpha-\alpha^2}\right) E\left(\tilde{\lambda}_{t+1}^{1-\alpha}\right) \right\}. \end{aligned}$$

Let $f_1(k) = k^{\alpha-\alpha^2}$ and $f_2(k) = k^{\alpha^2-\alpha^3}$.

Since both $f_1(k)$ and $f_2(k)$ are increasing functions of k , and $f_1(k)f_2(k) = k^{\alpha-\alpha^3}$,

by FKG or the Harris inequality, $E\left(\tilde{k}_t^{\alpha-\alpha^3}\right) \geq E\left(\tilde{k}_t^{\alpha-\alpha^2}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha^3}\right)$.

Similarly,

$$E\left(\tilde{\lambda}_t^{1-\alpha^2}\right) \geq E\left(\tilde{\lambda}_t^{1-\alpha}\right) E\left(\tilde{\lambda}_t^{\alpha-\alpha^2}\right).$$

$$\text{Thus, } \text{cov}(\tilde{p}_t^b, \tilde{p}_{t+1}^b) = \left[\frac{\beta E(\tilde{\lambda}^{-1})}{(\alpha\beta)^\alpha} \right]^2 (\alpha\beta)^{\alpha-\alpha^2} E\left(\tilde{\lambda}_{t+1}^{1-\alpha}\right) \geq 0$$

The inequality is strict if $\{\tilde{\lambda}_t\}$ is log-normally distributed. The proof in this case follows identically.

Proof of Proposition 3.2

$$\begin{aligned}
\text{a)} \quad & \text{cov}(\tilde{k}_t, \tilde{k}_{t+1}) \\
&= \text{cov}(\tilde{k}_t, i(\tilde{k}_t, \tilde{\lambda}_t) + (1 - \Omega)\tilde{k}_t) \\
&= \int \int (\tilde{k}_t, -\bar{k}) \left(i(\tilde{k}_t, \tilde{\lambda}_t) + (1 - \Omega)\tilde{k}_t - \bar{k} \right) dF(\tilde{k}_t) dG(\tilde{\lambda}_t) \\
&\text{Let } f_1(\tilde{k}_t, \tilde{\lambda}_t) = \tilde{k}_t - \bar{k}, \text{ and } f_2(\tilde{k}_t, \tilde{\lambda}_t) = i(\tilde{k}_t, \tilde{\lambda}_t) + (1 - \Omega)\tilde{k}_t - \bar{k}.
\end{aligned}$$

Both $f_1(\cdot)$ and $f_2(\cdot)$ are increasing functions of their arguments by assumption.

Hence by FKG or Harris inequality,

$$\begin{aligned}
& \int \int (\tilde{k}_t, -\bar{k}) \left(i(\tilde{k}_t, \tilde{\lambda}_t) + (1 - \Omega)\tilde{k}_t - \bar{k} \right) dF(\tilde{k}_t) dG(\tilde{\lambda}_t) \\
& \geq \int \int (\tilde{k}_t, -\bar{k}) dF(\tilde{k}_t) dG(\tilde{\lambda}_t) \times \int \int \left(i(\tilde{k}_t, \tilde{\lambda}_t) + (1 - \Omega)\tilde{k}_t - \bar{k} \right) dF(\tilde{k}_t) dG(\tilde{\lambda}_t) = 0
\end{aligned}$$

$$\text{Thus, } \text{cov}(p_{t-1}^e, p_t^e) = \text{cov}(\tilde{k}_t, \tilde{k}_{t+1}) \geq 0.$$

In the case of p_t^b ,

$$\begin{aligned}
& \text{cov}(p_t^b, p_{t+1}^b) = \text{cov}\left(h(\tilde{k}_t, \tilde{\lambda}_t), h(\tilde{k}_{t+1}, \tilde{\lambda}_{t+1})\right) \text{ where } h(\tilde{k}_t, \tilde{\lambda}_t) \text{ is given by (10).} \\
&= \text{cov}\left(-h(\tilde{k}_t, \tilde{\lambda}_t), -h\left(i(\tilde{k}_t, \tilde{\lambda}_t) + (1 - \Omega)\tilde{k}_t, \tilde{\lambda}_{t+1}\right)\right).
\end{aligned}$$

In general $h(\tilde{k}_t, \tilde{\lambda}_t)$ will be a decreasing function of each of its arguments, as it

is for specification (15). However, if $h(\tilde{k}_t, \tilde{\lambda}_t)$ is a decreasing function

$-h(\tilde{k}_t, \tilde{\lambda}_t)$ is increasing. Furthermore,

$$\text{cov}\left(h(\tilde{k}_t, \tilde{\lambda}_t), h(\tilde{k}_{t+1}, \tilde{\lambda}_{t+1})\right) = \text{cov}\left(-h(\tilde{k}_t, \tilde{\lambda}_t), -h(\tilde{k}_{t+1}, \tilde{\lambda}_{t+1})\right),$$

and the argument above may be applied.

Proof of Proposition 3.3

$$\begin{aligned}
\text{a)} \quad \text{cov}\left(\tilde{r}_t^e, \tilde{r}_{t+1}^e, \right) &= \text{cov}\left(\alpha \tilde{k}_t^{a-1} \tilde{\lambda}_t, \alpha \left[\alpha \beta \tilde{k}_t^a \tilde{\lambda}_t\right]^{\alpha-1} \tilde{\lambda}_{t+1}\right) \\
&= \alpha^2 (\alpha \beta)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right) \left\{ E\left(\tilde{k}_t^{a^2-1} \tilde{\lambda}_t^\alpha\right) - E\left(\tilde{k}_t^{\alpha-1} \tilde{\lambda}_t\right) E\left(\tilde{k}_t^{a^2-\alpha} \tilde{\lambda}_t^{\alpha-1}\right) \right\} \\
&\quad (\text{by independence of } \{\tilde{\lambda}_t\}). \\
&= \alpha^2 (\alpha \beta)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right) \left\{ E\left(\tilde{k}_t^{a^2-1}\right) E\left(\tilde{\lambda}_t^\alpha\right) - E\left(\tilde{k}_t^{\alpha-1}\right) E\left(\tilde{k}_t^{a^2-\alpha}\right) E\left(\tilde{\lambda}_t\right) E\left(\tilde{\lambda}_t^{\alpha-1}\right) \right\}. \quad (\text{TA 1})
\end{aligned}$$

We wish first to explore constituents of the preceding expression:

$$E\left(\tilde{k}_t^{a^2-1}\right) \text{ vs. } E\left(\tilde{k}_t^{a-1}\right) E\left(\tilde{k}_t^{a^2-\alpha}\right).$$

These expressions are of the general form

$$E\left(\tilde{k}_t^{\gamma_0+\gamma_1}\right) \text{ and } E\left(\tilde{k}_t^{\gamma_0}\right) E\left(\tilde{k}_t^{\gamma_1}\right) \text{ where } \gamma_0 < 0, \gamma_1 < 0.$$

$$\text{Define } \tilde{x}_t = \tilde{k}_t^{\gamma_0}, \text{ and } g\left(\tilde{x}_t\right) = \tilde{x}_t^{(\gamma_1/\gamma_0)}.$$

Since $(\gamma_1/\gamma_0) > 0$, $g\left(\tilde{x}_t\right)$ is an increasing function of \tilde{x}_t , and $g\left(\tilde{x}_t\right) = \tilde{k}_t^{\gamma_1}$.

$$\text{Thus, } E\left(\tilde{k}_t^{\gamma_0+\gamma_1}\right) = E\left(\tilde{x}_t g\left(\tilde{x}_t\right)\right) > E\left(\tilde{x}_t\right) E\left(g\left(\tilde{x}_t\right)\right) = E\left(\tilde{k}_t^{\gamma_0}\right) E\left(\tilde{k}_t^{\gamma_1}\right)$$

by the FKG or Harris inequality.

$$\text{Accordingly, } E\left(\tilde{k}_t^{a^2-1}\right) \geq E\left(\tilde{k}_t^{\alpha-1}\right) E\left(\tilde{k}_t^{a^2-\alpha}\right).$$

We may thus conclude that expression (TA 1) above is

$$\begin{aligned}
\text{cov}\left(\tilde{r}_t^e, \tilde{r}_{t+1}^e, \right) &\geq \alpha^2 (\alpha \beta)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right) \left\{ E\left(\tilde{k}_t^{a^2-1}\right) E\left(\tilde{\lambda}_t^\alpha\right) - E\left(\tilde{k}_t^{a^2-1}\right) E\left(\tilde{\lambda}_t\right) E\left(\tilde{\lambda}_t^{\alpha-1}\right) \right\} \\
&\geq \alpha^2 (\alpha \beta)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right) E\left(\tilde{k}_t^{a^2-1}\right) \left\{ E\left(\tilde{\lambda}_t^\alpha\right) - E\left(\tilde{\lambda}_t\right) E\left(\tilde{\lambda}_t^{\alpha-1}\right) \right\} \\
&\geq \alpha^2 (\alpha \beta)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right) E\left(\tilde{k}_t^{a^2-1}\right) \left\{ E\left(\tilde{\lambda}_t^\alpha\right) - E\left(\tilde{\lambda}_t\right) \left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha-1} \right\}
\end{aligned}$$

(by Jensen's inequality, since $\tilde{\lambda}^{\alpha-1}$, $0 < \alpha < 1$ is a convex function of $\tilde{\lambda}$)

$$\text{cov}\left(\tilde{r}_t^e, \tilde{r}_{t+1}^e, \right) \geq \alpha^2 (\alpha \beta)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right) E\left(\tilde{k}_t^{a^2-1}\right) \left\{ E\left(\tilde{\lambda}_t^\alpha\right) - \left(E\left(\tilde{\lambda}_t\right)\right)^\alpha \right\}$$

Again by Jensen's inequality, since $\tilde{\lambda}^\alpha$, $0 < \alpha < 1$ is a concave function of $\tilde{\lambda}$

$$E\left(\tilde{\lambda}_t^\alpha\right) - \left(E\left(\tilde{\lambda}_t\right)\right)^\alpha < 0$$

Hence, $\text{cov}\left(\tilde{r}_t^e, \tilde{r}_{t+1}^e\right) \geq -M$, $M > 0$. $\text{cov}\left(\tilde{r}_t^e, \tilde{r}_{t+1}^e\right)$ can be negative.

b) $\text{cov}\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) \geq 0$.

$$\begin{aligned} \text{cov}\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) &= \text{cov}\left(\frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)} \tilde{k}_t^{\alpha(\alpha-1)} \tilde{\lambda}_t^{\alpha-1}, \frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)} \tilde{k}_{t+1}^{\alpha(\alpha-1)} \tilde{\lambda}_{t+1}^{\alpha-1},\right) \\ &= \left(\frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)}\right)^2 \text{cov}\left(\tilde{k}_t^{\alpha(\alpha-1)} \tilde{\lambda}_t^{\alpha-1}, \left(\alpha\beta \tilde{k}_t^\alpha \tilde{\lambda}_t\right)^{\alpha(\alpha-1)} \tilde{\lambda}_{t+1}^{\alpha-1}\right) \\ &= (\alpha\beta)^{\alpha(\alpha-1)} \left(\frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)}\right)^2 \text{cov}\left(\tilde{k}_t^{\alpha^2-\alpha} \tilde{\lambda}_t^{\alpha-1}, \tilde{k}_t^{\alpha^3-\alpha^2} \tilde{\lambda}_t^{\alpha^2-\alpha} \tilde{\lambda}_{t+1}^{\alpha-1}\right) \end{aligned}$$

By the properties of the covariance function and that \tilde{k}_t , $\tilde{\lambda}_t$, and $\tilde{\lambda}_{t+1}$ are all independent of one another, the RHS expression becomes

$$= M E\left(\tilde{\lambda}_{t+1}^{\alpha-1}\right) \left\{ E\left(\tilde{k}_t^{\alpha^3-\alpha}\right) E\left(\tilde{\lambda}_t^{\alpha^2-1}\right) \right\} - E\left(\tilde{k}_t^{\alpha^2-\alpha}\right) E\left(\tilde{\lambda}_t^{\alpha-1}\right) E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\right) E\left(\tilde{\lambda}_t^{\alpha^2-\alpha}\right),$$

for some positive constant M . Let $f_1(\tilde{k}) = \tilde{k}^{\alpha^2-\alpha}$ and $f_2(\tilde{k}) = \tilde{k}^{\alpha^3-\alpha^2}$; each is a

decreasing function of k , furthermore, $f_1(\tilde{k})f_2(\tilde{k}) = \tilde{k}^{\alpha^3-\alpha}$ which is also

decreasing in k . By the FKG or Harris inequality

$$E\left(\tilde{k}_t^{\alpha^3-\alpha}\right) \geq E\left(\tilde{k}_t^{\alpha^2-\alpha}\right) E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\right) \text{ and}$$

$$E\left(\tilde{\lambda}_t^{\alpha^2-1}\right) \geq E\left(\tilde{\lambda}_t^{\alpha-1}\right) E\left(\tilde{\lambda}_t^{\alpha^2-\alpha}\right).$$

Thus $\text{cov}\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) \geq 0$.

We have been unable to derive any definitive result for the premium.

Proof of Proposition 4.1

This follows simply from the construction of the derived process $\{\tilde{\gamma}_t^{AB}\}$.

Every period the process $\{\tilde{\gamma}_t\}$ assumes a value above its mean, it is assuming a value in set γ^A . Furthermore, if the process $\{\tilde{\gamma}_t^{AB}\}$ is in set γ^A , then it is assuming a value above its mean. Thus the average number of periods the process $\{\tilde{\gamma}_t\}$ is above its mean ($\tilde{\gamma}_t \in \gamma^A$) must coincide with the average number of periods it is in set γ^A . Thus $ACT_{\{\tilde{\gamma}_t\}}^A = ACT_{\{\tilde{\gamma}_t^{AB}\}}^A$. A similar identification establishes that $ACT_{\{\tilde{\gamma}_t\}}^B = ACT_{\{\tilde{\gamma}_t^{AB}\}}^B$.

Properties of $\{\tilde{\gamma}^{AB}\}$

We first restrict attention to a consideration of arbitrary two state Markov chains. We adopt the convention that any measurement of the ‘time to crossing’ includes the crossing period itself.

A. Consider an arbitrary two state Markov chain with states γ_1, γ_2 and transition matrix:

$$\begin{matrix} & \gamma_1 & \gamma_2 \\ \begin{matrix} \gamma_1 \\ \gamma_2 \end{matrix} & \begin{bmatrix} \varphi_1 & 1 - \varphi_1 \\ 1 - \varphi_2 & \varphi_2 \end{bmatrix} \end{matrix}$$

where $0 < \varphi_1 < 1$ and $0 < \varphi_2 < 1$. The associated ergodic probability distribution (π_1, π_2) satisfies

$$[\pi_1, \pi_2] = [\pi_1, \pi_2] \begin{bmatrix} \varphi_1 & 1 - \varphi_1 \\ 1 - \varphi_2 & \varphi_2 \end{bmatrix},$$

with solution $\pi_1 = \frac{1 - \phi_2}{2 - (\phi_1 + \phi_2)}$ and $\pi_2 = \frac{1 - \phi_1}{2 - (\phi_1 + \phi_2)}$

B. Suppose the process is in state $\tilde{\gamma} = \gamma_1$. The average time to crossing to state 2, ACT_1 , is given by:

$$\begin{aligned} ACT_1 &= \sum_{n=1}^{\infty} n \text{Prob} \left(\begin{array}{c} \tilde{\gamma} = \gamma_2 \\ \text{in step } n \end{array} \mid \begin{array}{c} \tilde{\gamma} = \gamma_1, \text{ for} \\ \text{the } n-1 \text{ prior steps} \end{array} \right) \\ &= \sum_{n=1}^{\infty} n \varphi_1^{n-1} (1 - \varphi_1) = (1 - \varphi_1) \sum_{n=1}^{\infty} n \varphi_1^{n-1} = (1 - \varphi_1) \left(\frac{1}{(1 - \varphi_1)^2} \right) = \frac{1}{1 - \varphi_1} \end{aligned}$$

Similarly, $ACT_2 = \frac{1}{1 - \varphi_2}$

.

Accordingly, the average crossing time, ACT , satisfies

$$\begin{aligned} ACT &= \pi_1 ACT_1 + \pi_2 ACT_2 = \pi_1 \left(\frac{1}{1 - \varphi_1} \right) + \pi_2 \left(\frac{1}{1 - \varphi_2} \right) \\ &= \frac{1}{2 - (\phi_1 + \phi_2)} \left[\frac{1 - \phi_2}{1 - \phi_1} + \frac{1 - \phi_1}{1 - \phi_2} \right] \end{aligned}$$

$$\begin{aligned} \text{C. We compute } \text{corr}(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}) &= \frac{\text{cov}(\tilde{\gamma}_t, \tilde{\gamma}_{t+1})}{\sigma_{\tilde{\gamma}_t} \sigma_{\tilde{\gamma}_{t+1}}} \\ &= \frac{E(\tilde{\gamma}_t \tilde{\gamma}_{t+1}) - E(\tilde{\gamma}_t)E(\tilde{\gamma}_{t+1})}{\sigma_{\tilde{\gamma}_t} \sigma_{\tilde{\gamma}_{t+1}}} \end{aligned}$$

Without loss of generality, we assume $\gamma_1 = 1$ and $\gamma_2 = -1$, since the $\text{corr}(\tilde{\gamma}_t, \tilde{\gamma}_{t+1})$ is determined by the structure of the transition matrix and not the specific values assumed by $\tilde{\gamma}_t$.²

² Consider a stochastic process $\tilde{\gamma}_t$; then for any $a > 0$ and any $b \in \mathbb{R}$, $\text{corr}(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}) = \text{corr}(a\tilde{\gamma}_t + b, a\tilde{\gamma}_{t+1} + b)$.

The correlation computation requires the following constituents:

$$\begin{aligned}
\text{(i)} \quad E(\tilde{\gamma}_t) &= \gamma_1 \left(\frac{1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right) + \gamma_2 \left(\frac{1 - \phi_1}{2 - (\phi_1 + \phi_2)} \right) \\
&= \frac{\phi_1 - \phi_2}{2 - (\phi_1 + \phi_2)} \\
\text{(ii)} \quad E(\tilde{\gamma}_{t+1}) &= \left(\frac{\phi_1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right) \\
\text{(iii)} \quad \text{var}(\tilde{\gamma}_t) &= E(\tilde{\gamma}_t - E(\tilde{\gamma}_t))^2 \\
&= \left(\frac{1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right) \left(1 - \frac{\phi_1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right)^2 \\
&\quad + \left(\frac{1 - \phi_1}{2 - (\phi_1 + \phi_2)} \right) \left(-1 - \frac{\phi_1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right)^2 \\
&= \left(\frac{1}{2 - (\phi_1 + \phi_2)} \right)^3 \left[(1 - \phi_2)(2 - 2\phi_1)^2 + (1 - \phi_1)(-2 + 2\phi_2)^2 \right] \\
&= \frac{4(1 - \phi_1)(1 - \phi_2)}{[2 - (\phi_1 + \phi_2)]^3} \{2 - (\phi_1 + \phi_2)\} \\
&= \frac{4(1 - \phi_1)(1 - \phi_2)}{[2 - (\phi_1 + \phi_2)]^2} = \text{var}(\tilde{\gamma}_{t+1})
\end{aligned}$$

We can thus assume $\gamma_1 = 1$ and $\gamma_2 = -1$, if constants a, b as per the above correlation equality satisfy

$a\gamma_1 + b = 1$ and $a\gamma_2 + b = -1$. The solution to this simple system of equations is $a = \frac{2}{\gamma_1 - \gamma_2}$ and

$$b = (-1 + \gamma_2 / \gamma_1) / (1 - \gamma_2 / \gamma_1).$$

$$\begin{aligned}
\text{(iv)} \quad E(\tilde{\gamma}_t \tilde{\gamma}_{t+1}) &= \sum_{\gamma_t} \text{Prob}(\gamma_t) \left[\sum_{\gamma_{t+1}} \gamma_{t+1} \text{Prob}(\gamma_{t+1} | \gamma_t) \right] (\gamma_t) \\
&= 1 \left[\frac{1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right] [1\phi_1 + (1 - \phi_1)(-1)] \\
&\quad + (-1) \left[\frac{1 - \phi_1}{2 - (\phi_1 + \phi_2)} \right] [1(1 - \phi_2)(-1\phi_2)] \\
&= \frac{1}{2 - (\phi_1 + \phi_2)} \{ (1 - \phi_2)[-1 + 2\phi_1] - (1 - \phi_1)[1 - 2\phi_2] \} \\
&= \frac{1}{2 - (\phi_1 + \phi_2)} \{ -2 + 3\phi_1 + 3\phi_2 - 4\phi_1\phi_2 \}
\end{aligned}$$

$$\begin{aligned}
\text{(v)} \quad \text{corr}(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}) &= \frac{E(\tilde{\gamma}_t \tilde{\gamma}_{t+1}) - E(\tilde{\gamma}_t)E(\tilde{\gamma}_{t+1})}{\sigma_{\tilde{\gamma}_t} \sigma_{\tilde{\gamma}_{t+1}}} \\
&= \frac{\frac{1}{2 - (\phi_1 + \phi_2)} [-2 + 3(\phi_1 + \phi_2) - 4\phi_1\phi_2] - \left[\frac{\phi_1 - \phi_2}{2 - (\phi_1 + \phi_2)} \right]^2}{\frac{4(1 - \phi_1)(1 - \phi_2)}{(2 - (\phi_1 + \phi_2))^2}} \\
&= \frac{[2 - (\phi_1 + \phi_2)][-2 + 3(\phi_1 + \phi_2) - 4\phi_1\phi_2] - [\phi_1 - \phi_2]^2}{4(1 - \phi_1)(1 - \phi_2)}
\end{aligned}$$

(after considerable simplification)

$$\begin{aligned}
&= \frac{-4 + (8 + 4\phi_1\phi_2)[\phi_1 + \phi_2] - 12\phi_1\phi_2 - 4\phi_1^2 - 4\phi_2^2}{4(1 - \phi_1)(1 - \phi_2)} \\
&= \frac{-(1 + \phi_1\phi_2) + (\phi_1 + \phi_2)[1 + (1 - \phi_1)(1 - \phi_2)]}{(1 - \phi_1)(1 - \phi_2)} \\
&= (\phi_1 + \phi_2) + \left[\frac{-(1 - \phi_1)(1 - \phi_2)}{(1 - \phi_1)(1 - \phi_2)} \right] = (\phi_1 + \phi_2) - 1 \tag{A}
\end{aligned}$$

D. Proposition 4.2. To show $ACT_{\{\tilde{\gamma}_t\}} \leq 2$ implies $corr(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}) \leq 0$.

We show $ACT \leq 2$ implies $(\phi_1 + \phi_2) \leq 1$.

$$ACT_{\{\tilde{\gamma}_t\}} \leq 2 \Rightarrow$$

$$\frac{1}{2 - (\phi_1 + \phi_2)} \left[\frac{1 - \phi_1}{1 - \phi_2} + \frac{1 - \phi_2}{1 - \phi_1} \right] \leq 2$$

The second term in the ACT expression is of the form $x + \frac{1}{x}$, which assumes

a minimum at $x = 1$. Therefore,

$$\frac{1}{2 - (\phi_1 + \phi_2)} [2] \leq ACT_{\{\tilde{\gamma}_t\}} \leq 2, \text{ or}$$

$$\frac{2}{2 - (\phi_1 + \phi_2)} \leq 2; \text{ equivalently}$$

$$1 \leq 2 - (\phi_1 + \phi_2) \text{ or}$$

$$(\phi_1 + \phi_2) \leq 1.$$

Suppose $\phi_1 = \phi_2$ and $(\phi_1 + \phi_2) \leq 1$.

$$\text{Then } ACT_{\{\tilde{\gamma}_t\}} = \frac{1}{2 - (\phi_1 + \phi_2)} \left[\frac{1 - \phi_1}{1 - \phi_2} + \frac{1 - \phi_2}{1 - \phi_1} \right] \leq \left[\frac{1 - \phi_1}{1 - \phi_2} + \frac{1 - \phi_2}{1 - \phi_1} \right] = 2$$

E. When $\phi_1 = \phi_2 = \phi$, the ACT reduces to

$$ACT = \frac{1}{2 - 2\phi} [1 + 1] = \frac{1}{1 - \phi}, \text{ and}$$

$$corr(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}) = 2\phi - 1. \text{ Thus}$$

$$\text{corr}(\gamma_t^x, \gamma_{t+1}^x) \geq \text{corr}(\gamma_t^y, \gamma_{t+1}^y), \text{ if and only if}$$

$$2\phi^x - 1 > 2\phi^y - 1, \text{ if and only if}$$

$$\phi^x > \phi^y, \text{ if and only if}$$

$$\frac{1}{1-\phi^x} > \frac{1}{1-\phi^y}, \text{ if and only if}$$

$$ACT_{\{\gamma_t^x\}} > ACT_{\{\gamma_t^y\}}.$$

F. The region A is computed by searching for the pairs $(0, 0) < (\phi_1, \phi_2) < (1, 1)$ such that

$$\partial ACT / \partial \phi_1 > 0 \text{ and } \partial ACT / \partial \phi_2 > 0.$$

The indicated region (Figure 2) was constructed numerically.

Proof of Proposition 5.1

(a) We first offer the proof for $\{\tilde{p}_t^e\}; \{\tilde{d}_t\}$ is analyzed similarly, since

$$d_t = ((1 - \beta) / \beta) p_t^e$$

$$\begin{aligned} \text{cov}(\tilde{p}_t^e, \tilde{p}_{t+1}^e) &= \text{cov}(\tilde{k}_{t+1}, \tilde{k}_{t+2}) \\ &= \text{cov}\left(\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, \alpha\beta\left[\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}\right]^\alpha e^{\tilde{\lambda}_{t+1}}\right) \\ &= \text{cov}\left(\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, (\alpha\beta)^{1+\alpha} \tilde{k}_t^{\alpha^2} e^{\alpha\tilde{\lambda}_t} e^{(\rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1})}\right) \\ &= (\alpha\beta)^{2+\alpha} \text{cov}\left(\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^2} e^{(\alpha+\rho)\tilde{\lambda}_t} \cdot e^{\tilde{\varepsilon}_{t+1}}\right) \\ &= (\alpha\beta)^{2+\alpha} E\left(e^{\tilde{\varepsilon}_{t+1}}\right) \left\{ E\left(\tilde{k}_t^{\alpha+\alpha^2} e^{(\alpha+\rho+1)\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2} e^{(\alpha+\rho)\tilde{\lambda}_t}\right) \right\}. \end{aligned}$$

$$\text{Let } g^1(k_t, \lambda_t) = \tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, g_1^1 > 0; g_2^1 > 0$$

$$g^2(k_t, \lambda_t) = \tilde{k}_t^{\alpha^2} e^{(\alpha+\rho)\tilde{\lambda}_t}, \quad g_1^2 > 0; \quad g_2^2 > 0$$

By FKG or the Harris inequality

$$\begin{aligned} E(g^1(\tilde{k}_t, \tilde{\lambda}_t) g^2(\tilde{k}_t, \tilde{\lambda}_t)) \\ \geq E(g^1(\tilde{k}_t, \tilde{\lambda}_t)) E(g^2(\tilde{k}_t, \tilde{\lambda}_t)) \end{aligned}$$

$$\text{Thus, } \text{cov}(\tilde{p}_t^e, \tilde{p}_{t+1}^e) = \text{cov}(\tilde{k}_{t+1}^e, \tilde{k}_{t+2}^e) \geq 0.$$

$$(b) \quad \text{cov}(\tilde{p}_t^b, \tilde{p}_{t+1}^b)$$

$$\text{where } p_t^b = \beta e^{\sigma_\varepsilon^2/2} (\alpha\beta)^{-2} k_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\tilde{\lambda}_t}$$

$$\text{cov}(\tilde{p}_t^b, \tilde{p}_{t+1}^b) = \beta^2 e^{\sigma_\varepsilon^2} (\alpha\beta)^{-2\alpha} \text{cov}\left(\tilde{k}_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\tilde{\lambda}_t}, \left(\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}\right)^{\alpha-\alpha^2} e^{(1-\alpha-\rho)(\rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1})}\right)$$

$$= \beta^2 e^{\sigma_\varepsilon^2} (\alpha\beta)^{-2\alpha} (\alpha\beta)^{\alpha-\alpha^2} \text{cov}\left(\tilde{k}_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^2-\alpha^3} e^{(\alpha-\alpha^2)\tilde{\lambda}_t} e^{(1-\alpha-\rho)(\rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1})}\right)$$

$$\underbrace{\hspace{10em}}_M$$

$$= M \left(\text{cov} \tilde{k}_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^2-\alpha^3} e^{(\alpha-\alpha^2+\rho(1-\alpha-\rho))\tilde{\lambda}_t} e^{(1-\alpha-\rho)\tilde{\varepsilon}_{t+1}} \right)$$

$$= M e^{(1-\alpha-\rho)^2 \sigma_\varepsilon^2/2} \left\{ E \left(\tilde{k}_t^{\alpha-\alpha^3} e^{(\alpha-\alpha^2+(1+\rho)(1-\alpha-\rho))\tilde{\lambda}_t} \right) - E \left(\tilde{k}_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\tilde{\lambda}_t} \right) E \left(\tilde{k}_t^{\alpha^2-\alpha^3} e^{(\alpha-\alpha^2+\rho(1-\alpha-\rho))\tilde{\lambda}_t} \right) \right\}$$

$$\text{If we let } g(\tilde{k}_t, \tilde{\lambda}_t) = \tilde{k}_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\tilde{\lambda}_t}$$

and

$$f(\tilde{k}_t, \tilde{\lambda}_t) = \tilde{k}_t^{\alpha^2-\alpha^3} e^{(\alpha-\alpha^2+\rho(1-\alpha-\rho))\tilde{\lambda}_t},$$

then, if $\alpha + \rho < 1$,

$$f_1(\quad) > 0 \quad f_2(\quad) > 0, \quad g_1(\quad) > 0 \quad g_2(\quad) > 0.$$

By the FKG or the Harris inequality we conclude immediately that

$$\text{cov}\left(\tilde{p}_t^b, \tilde{p}_{t+1}^b\right) \geq 0, \text{ provided } \alpha + \rho < 1.$$

Proof of Proposition 5.2

$$(a) \quad \text{cov}\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) = \text{cov}\left(\tilde{r}_{t+1}^b, \tilde{r}_{t+2}^b\right)$$

$$= \underbrace{\left[\frac{(\alpha\beta)^\alpha}{\beta} e^{-\sigma_\varepsilon^2/2} \right]^2}_L \text{cov}\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t}, \tilde{k}_{t+1}^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_{t+1}}\right)$$

But $\tilde{k}_{t+1} = \alpha\beta k_t^\alpha e^{\tilde{\lambda}_t}$ and $\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}$; thus:

$$\begin{aligned} &= L(\alpha\beta)^{\alpha^2-\alpha} \text{cov}\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^3-\alpha^2} e^{\tilde{\lambda}_t(\alpha^2-\alpha)} e^{(\alpha+\rho-1)\rho\tilde{\lambda}_t + (\alpha+\rho-1)\tilde{\varepsilon}_{t+1}}\right) \\ &= L(\alpha\beta)^{\alpha^2-\alpha} \text{cov}\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+(\alpha+\rho-1)\rho]\tilde{\lambda}_t} \cdot e^{(\alpha+\rho-1)\tilde{\varepsilon}_{t+1}}\right) \\ &= L(\alpha\beta)^{\alpha^2-\alpha} e^{(\alpha+\rho-1)^2\sigma_\varepsilon^2/2} \left\{ E\left(\tilde{k}_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+2(\alpha+\rho-1)]\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+(\alpha+\rho-1)\rho]\tilde{\lambda}_t}\right) \right\} \end{aligned}$$

$$\text{Let } g^1(\tilde{k}, \tilde{\lambda}) = \left(\tilde{k}_t^{\alpha^2-\alpha}\right) e^{(\alpha+\rho-1)\tilde{\lambda}_t}$$

$$g^2(\tilde{k}, \tilde{\lambda}) = \tilde{k}_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+(\alpha+\rho-1)\rho]\tilde{\lambda}_t}$$

$$g_1^1(\tilde{k}, \tilde{\lambda}) < 0 \quad g_2^1(\tilde{k}, \tilde{\lambda}) < 0 \quad \text{if } \alpha + \rho < 1$$

$$g_1^2(\tilde{k}, \tilde{\lambda}) < 0 \quad g_2^2(\tilde{k}, \tilde{\lambda}) < 0 \quad \text{if } \alpha + \rho < 1.$$

Hence, $-g^1(\tilde{k}_t, \tilde{\lambda}_t)$ is increasing

and $-g^2(\tilde{k}_t, \tilde{\lambda}_t)$ is increasing. Thus,

$$\int -g^1(\tilde{k}_t, \tilde{\lambda}_t) \left(-g^2(\tilde{k}_t, \tilde{\lambda}_t)\right) dF(\tilde{k}_t, \tilde{\lambda}_t) \geq \int -g^1(\tilde{k}_t, \tilde{\lambda}_t) dF(\tilde{k}_t, \tilde{\lambda}_t) \int -g^2(\tilde{k}_t, \tilde{\lambda}_t) dF(\tilde{k}_t, \tilde{\lambda}_t)$$

$$\int g^1(\tilde{k}_t, \tilde{\lambda}_t) g^2(\tilde{k}_t, \tilde{\lambda}_t) dF(\tilde{k}_t, \tilde{\lambda}_t) \geq \int g^1(\tilde{k}_t, \tilde{\lambda}_t) dF(\tilde{k}_t, \tilde{\lambda}_t) \int g^2(\tilde{k}_t, \tilde{\lambda}_t) dF(\tilde{k}_t, \tilde{\lambda}_t),$$

(by the FKG or the Harris inequality).

$$\text{Thus, } \text{cov}(\tilde{r}_t^b, \tilde{r}_{t+1}^b) \geq 0.$$

$$\begin{aligned} \text{(b)} \quad & \text{cov}(\tilde{r}_t^e, \tilde{r}_{t+1}^e) \\ &= \text{cov}\left(\alpha \tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}, \alpha \tilde{k}_{t+1}^{\alpha-1} e^{\tilde{\lambda}_{t+1}}\right) \\ &= \text{cov}\left(\alpha \tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}, \alpha \left(\alpha \beta \tilde{k}_t^\alpha e^{\tilde{\lambda}_t}\right)^{\alpha-1} e^{\rho \tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}}\right) \\ &= \text{cov}\left(\alpha \tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}, \alpha^\alpha \beta^{\alpha-1} \tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha-1)\tilde{\lambda}_t} e^{\rho \tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}}\right) \\ &= \underbrace{\alpha^{1+\alpha} \beta^{\alpha-1}}_{M > 0} \text{cov}\left(\tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t} e^{\tilde{\varepsilon}_{t+1}}\right) \\ &= M \left\{ E\left(\tilde{k}_t^{\alpha^2-1} e^{(\alpha+\rho)\tilde{\lambda}_t} e^{\tilde{\varepsilon}_{t+1}}\right) - E\left(\tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t} e^{\tilde{\varepsilon}_{t+1}}\right) \right\} \\ &= M e^{\sigma_\varepsilon^2/2} \left\{ E\left(\tilde{k}_t^{\alpha^2-1} e^{(\alpha+\rho)\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right) \right\} \end{aligned} \quad (\text{TA } 2)$$

(i) From the proof of Proposition 3.3,

$$E\left(\tilde{k}_t^{\alpha^2-1}\right) > E\left(\tilde{k}_t^{\alpha-1}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha}\right).$$

(ii) If $(\alpha + \rho) > 1$, then $e^{(\alpha+\rho)\tilde{\lambda}_t}$, $e^{\tilde{\lambda}_t}$ and $e^{(\alpha+\rho-1)\tilde{\lambda}_t}$ are all increasing functions of

$\tilde{\lambda}_t$. Thus, by the FKG or the Harris inequality,

$$E\left(e^{(\alpha+\rho)\tilde{\lambda}_t}\right) \geq E\left(e^{\tilde{\lambda}_t}\right) E\left(e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right).$$

Since (TA 2) is equivalent to (TA 3)

$$= M e^{\sigma_\varepsilon^2/2} \left\{ E\left(\tilde{k}_t^{\alpha^2-1}\right) E\left(e^{(\alpha+\rho)\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha-1}\right) E\left(e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha}\right) E\left(e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right) \right\} \quad (\text{TA } 3)$$

then by relationship (i), (ii) noted previously, we have

$$\text{cov}(\tilde{r}_t^e, \tilde{r}_{t+1}^e) \geq 0, \text{ provided } (\alpha + \rho) > 1.$$

Proof of Proposition 5.3 (This result is due to Sergio Villar)

This proof uses the fact that if $\tilde{y} \sim N(0, v)$, then $E[\exp(\tilde{y})] = \frac{v}{2}$

For the $AR(1)$ process, $\text{Var}(\tilde{x}_t) = \sigma^2 \left(\frac{1 - \rho^{2t}}{1 - \rho^2} \right) :$

$$\begin{aligned} \text{Cov}(\tilde{\lambda}_t, \tilde{\lambda}_{t+1}) &= \text{Cov}(\exp(\tilde{x}_t), \exp(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1})) \\ &= E[\exp(\tilde{x}_t)\exp(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1})] - E[\exp(\tilde{x}_t)]E[\exp(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1})] \\ &= E[\exp((\rho + 1)\tilde{x}_t + \tilde{\varepsilon}_{t+1})] - E[\exp(\tilde{x}_t)]E[\exp(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1})] \\ &= E[\exp((\rho + 1)\tilde{x}_t)]E[\exp(\tilde{\varepsilon}_{t+1})] - E[\exp(\tilde{x}_t)]E[\exp(\rho\tilde{x}_t)]E[\exp(\tilde{\varepsilon}_{t+1})] \\ &= E[\exp(\tilde{\varepsilon}_{t+1})]\left(E[\exp((\rho + 1)\tilde{x}_t)] - E[\exp(\tilde{x}_t)]E[\exp(\rho\tilde{x}_t)]\right) \\ &= \exp\left(\frac{\sigma^2}{2}\right)\left(\exp\left(\frac{1}{2}(\rho + 1)^2\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right) - \exp\left(\frac{1}{2}\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\exp\left(\frac{1}{2}\rho^2\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\right) \\ &= \exp\left(\frac{\sigma^2}{2}\right)\left(\exp\left(\frac{1}{2}(\rho^2 + 2\rho + 1)\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right) - \exp\left(\frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\right) \\ &= \exp\left(\frac{\sigma^2}{2}\right)\exp\left(\frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right) - 1\right) \\ &= \exp\left(\frac{\sigma^2}{2} + \frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right) - 1\right) \\ &= \exp\left(\frac{\sigma^2}{2}\left(1 + (\rho^2 + 1)\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right) - 1\right) \end{aligned}$$

$$\text{Thus } \text{Cov}(\tilde{\lambda}_t, \tilde{\lambda}_{t+1}) = \exp\left(\frac{\sigma^2}{2}\left(1 + (\rho^2 + 1)\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right)\right) - 1\right).$$

Now, clearly, the first element is positive since it is an exponent. Further, since

$$\sigma^2\left(\frac{1 - \rho^{2t}}{1 - \rho^2}\right) > 0,$$

we know that the second element is a strictly increasing function of ρ , reaching a value of zero at $\rho = 0$. Therefore, for $\rho < 0$, the expression is negative, while for $\rho > 0$ the expression is positive.