A Appendix

A.1 Proofs

Derivation of identified set in (2.2). Following Uhlig (2005), we reparameterize A via the Cholesky matrix Σ_{tr} and a rotation matrix $Q = \begin{pmatrix} cos\rho & -sin\rho \\ sin\rho & cos\rho \end{pmatrix}$ with spherical coordinate $\rho \in [0, 2\pi]$. We can then express α as a function of ϕ and the non-identified parameter ρ indexing a rotation matrix;

$$A^{-1} = \Sigma_{tr} Q = \begin{pmatrix} \sigma_{11}\cos\rho & -\sigma_{11}\sin\rho \\ \sigma_{21}\cos\rho + \sigma_{22}\sin\rho & -\sigma_{21}\sin\rho + \sigma_{22}\cos\rho \end{pmatrix}$$

and the parameter of interest is $\alpha = \alpha(\rho, \phi) \equiv \sigma_{11}\cos\rho$. We impose the sign normalization restrictions throughout by constraining the diagonal elements of A to being nonnegative,

$$\sigma_{22}\cos\rho - \sigma_{21}\sin\rho \geq 0 \text{ and } \sigma_{11}\cos\rho \geq 0. \tag{A.1}$$

The sign restrictions $a_{12} \geq 0$ and $a_{21} \leq 0$ are expressed as

$$\sigma_{11}\sin\rho \geq 0 \tag{A.2}$$

$$-\sigma_{22}\sin\rho - \sigma_{21}\cos\rho \leq 0. \tag{A.3}$$

Given ϕ, the identified set for $\alpha = \sigma_{11}\cos\rho$ is given by its set as ρ varies over the set characterized by the restrictions (A.1) - (A.3). Since the second constraint in (A.1) and (A.2) imply $\rho \in [0, \pi/2]$, we focus on how the other two restrictions (the first constraint in (A.1) and (A.3)) tighten up $\rho \in [0, \pi/2]$.

Assume $\sigma_{21} > 0$. Then, they imply $\arctan(-\sigma_{21}/\sigma_{22}) \leq \rho \leq \arctan(\sigma_{22}/\sigma_{21})$. Intersecting this interval with $\rho \in [0, \pi/2]$ leads to $[0, \arctan(\sigma_{22}/\sigma_{21})]$ as the identified set for ρ. Hence, the identified set for α in the $\sigma_{21} > 0$ case follows. A similar argument leads to the α identified set for the $\sigma_{21} \leq 0$ case. ■

Proof of Lemma 3.1. (i) By the construction of the ϕ-prior (3.2), the marginal likelihood for $M \in \mathcal{M}^s$ can be rewritten as

$$\begin{aligned} p(Y|M) &= \int_\Phi p(Y|\phi, M) d\pi_{\phi|M}(\phi) = \int_\Phi p(Y|\phi) \cdot \frac{1\{IS_\alpha(\phi|M) \neq \emptyset\}}{\tilde{\pi}_\phi(IS_\alpha(\phi|M) \neq \emptyset)} d\tilde{\pi}_\phi(\phi) \\ &= \tilde{p}(Y) \int_\phi \frac{1\{IS_\alpha(\phi|M) \neq \emptyset\}}{\tilde{\pi}_\phi(IS_\alpha(\phi|M) \neq \emptyset)} d\tilde{\pi}_{\phi|Y}(\phi) = \tilde{p}(Y) \frac{\tilde{\pi}_{\phi|Y}(IS_r(\phi|M) \neq \emptyset)}{\tilde{\pi}_\phi(IS_r(\phi|M) \neq \emptyset)} = \tilde{p}(Y) O_M, \end{aligned}$$

where the second line uses the assumption that the set-identified models admit an identical reduced-form and the third line follows from the Bayes theorem for the reduced-form parameters, $p(Y|\phi)\tilde{\pi}_\phi(\phi) = \tilde{p}(Y)\tilde{\pi}_{\phi|Y}(\phi)$. Plugging this expression of the marginal likelihood into (3.1) leads to the claim.

(ii) Under the additionally imposed assumptions, the marginal likelihood of model $M^p \in \mathcal{M}^p$ is given by $\tilde{p}(Y)O_{M^p}$. Hence, combined with $p(Y|M^s) = \tilde{p}(Y)O_{M^s}$ shown in part (i), (3.5) follows.

(iii) The claim follows immediately by noting that the imposed assumptions imply $O_M = 1$ for all $M \in \mathcal{M}$ and setting O_M, $M \in \mathcal{M}$, to one in (3.5). ■

Derivation of $\Pi_{\alpha|M^s,Y}$ in equation (3.7). We derive $\Pi_{\alpha|M^s,Y}$ in the next lemma:

Lemma A.1 *For a set-identified model M^s with the structural parameters $\theta_{M^s} \in \Theta_{M^s}$ and reduced-form parameters $\phi_{M^s} = g_{M^s}(\theta_{M^s}) \in \Phi_{M^s} = g_{M^s}(\Theta_{M^s})$, let a prior for ϕ_{M^s}, $\pi_{\phi_{M^s}|M^s}$ be given. Define the class of priors of θ_{M^s} by*

$$\Pi_{\theta_{M^s}|M^s} \equiv \left\{\pi_{\theta_{M^s}|M^s} : \pi_{\theta_{M^s}|M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(B)) = \pi_{\phi_{M^s}|M^s}(B),\ \forall B \in \mathcal{B}(\Phi_{M^s})\right\}.$$

Updating $\Pi_{\theta_{M^s}|M^s}$ prior-by-prior with the likelihood $\tilde{p}(Y|\theta_{M^s}, M^s)$ and marginalizing the resulting posteriors via $\alpha = \alpha_{M^s}(\theta_{M^s})$ leads to the following set of posteriors for α:

$$\Pi_{\alpha|M^s,Y} \equiv \left\{\pi_{\alpha|M^s,Y} = \int_{\Phi_M} \pi_{\alpha|M^s,\phi_{M^s}} d\pi_{\phi_{M^s}|M^s,Y} : \pi_{\alpha|M^s,\phi_{M^s}}(IS_\alpha(\phi_{M^s}|M^s)) = 1,\ \pi_{\phi_{M^s}|M^s}\text{-}a.s.\right\}. \tag{A.4}$$

■

Proof of Lemma A.1. The prior-by-prior updating rule updates $\Pi_{\theta_{M^s}|M^s}$ to

$$\Pi_{\theta_{M^s}|M^s,Y} \equiv \left\{\pi_{\theta_{M^s}|M^s,Y} : \pi_{\theta_{M^s}|M^s,Y}(\Theta_{M^s} \cap g_{M^s}^{-1}(B)) = \pi_{\phi_{M^s}|M^s,Y}(B),\ \forall B \in \mathcal{B}(\Phi_{M^s})\right\}.$$

Since $\pi_{\theta_{M^s}|M^s,Y}(\Theta_{M^s} \cap g_{M^s}^{-1}(B))$ can be written as

$$\pi_{\theta_{M^s}|M^s,Y}(\Theta_{M^s} \cap g_{M^s}^{-1}(B)) = \int_B \pi_{\theta_{M^s}|\phi_{M^s},M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(\phi_{M^s}))d\pi_{\phi_{M^s}|M^s,Y}(\phi_{M^s}),$$

the ϕ_{M^s}-marginal constraints for $\pi_{\theta_{M^s}|M^s,Y}$ are equivalent to

$$\int_B \pi_{\theta_{M^s}|\phi_{M^s},M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(\phi_{M^s}))d\pi_{\phi_{M^s}|M^s,Y}(\phi_{M^s}) = \pi_{\phi_{M^s}|M^s,Y}(B).$$

This equality holds for all $B \in \mathcal{B}(\Phi_{M^s})$ if and only if $\pi_{\theta_{M^s}|\phi_{M^s},M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(\phi_{M^s})) = 1$, $\pi_{\phi_{M^s}|M^s,Y}$-a.s. Accordingly, an equivalent representation of the class of posteriors for θ_{M^s} is

$$\Pi_{\theta_{M^s}|M^s,Y} = \left\{\int_{\Phi_{M^s}} \pi_{\theta_{M^s}|\phi_{M^s},M^s} d\pi_{\Phi_{M^s}|Y} : \pi_{\theta_{M^s}|\phi_{M^s},M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(\phi_{M^s})) = 1, \pi_{\phi_{M^s}|M^s,Y}\text{-a.s.}\right\}. \tag{A.5}$$

Note that we have

$$\pi_{\alpha|\phi_{M^s},M^s}(IS_\alpha(\phi_{M^s}|M^s)) = \pi_{\theta_{M^s}|\phi_{M^s},M^s}(\alpha_{M^s}^{-1}(IS_\alpha(\phi_{M^s}|M^s))) \\ = \pi_{\theta_{M^s}|\phi_{M^s},M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(\phi_{M^s})),$$

where the second equality follows by the definition of the identified set of α. Hence, $\pi_{\theta_{M^s}|\phi_{M^s},M^s}(\Theta_{M^s} \cap g_{M^s}^{-1}(\phi_{M^s})) = 1$, $\pi_{\phi_{M^s}|M^s,Y}$-a.s. holds if and only if $\pi_{\alpha|\phi_{M^s},M^s}(IS_\alpha(\phi_{M^s}|M^s)) = 1$, $\pi_{\phi_{M^s}|M^s,Y}$-a.s. The class of marginalized posteriors for α (A.4) therefore follows. ■

Proof of Proposition 3.1. Let $\pi_{\theta,M}$ be a prior of (θ, M) belonging to the proposed $\Pi_{\theta,M}$. The corresponding posterior for θ with M integrated out can be computed as follows: for any measurable subset $H \subset \Theta$,

$$\begin{aligned}
\pi_{\theta|Y}(H) &= \frac{\sum_{M\in\mathcal{M}} \int_H \tilde{p}(Y|\theta, M) d\pi_{\theta|M}(\theta)\pi_M}{\sum_{M\in\mathcal{M}} \left[\int_{\Theta_M} \tilde{p}(Y|\theta, M) d\pi_{\theta|M}(\theta)\right] \pi_M} \\
&= \frac{\begin{pmatrix} \sum_{M^p\in\mathcal{M}^p} \pi_{\theta|M^p,Y}(H)p(Y|M^p)\pi_{M^p} \\ + \sum_{M^s\in\mathcal{M}^s} \left[\int_{\Phi_{M^s}} \pi_{\theta|\phi_{M^s},M^s}(H)p(Y|\phi_{M^s}, M^s)d\pi_{\phi_{M^s}|M^s}(\phi_{M^s})\right] \pi_{M^s} \end{pmatrix}}{\sum_{M^p\in\mathcal{M}^p} p(Y|M^p)\pi_{M^p} + \sum_{M^s\in\mathcal{M}^s} \left[\int_{\Phi_{M^s}} p(Y|\phi_{M^s}, M^s)d\pi_{\phi_{M^s}|M^s}(\phi_{M^s})\right] \pi_{M^s}} \\
&= \sum_{M^p\in\mathcal{M}^p} \pi_{\theta|M^p}(H)\pi_{M^p|Y} + \sum_{M^s\in\mathcal{M}^s} \left[\int_{\Phi_{M^s}} \pi_{\theta|\phi_{M^s},M^s}(H)d\pi_{\phi_{M^s}|M^s,Y}(\phi_{M^s})\right] \pi_{M^s|Y}
\end{aligned}$$

where the second line uses

$$\begin{aligned}
\int_H \tilde{p}(Y|\theta, M)d\pi_{\theta|M}(\theta) &= \int_{\Phi_M} \left[\int_\Theta 1\{\theta \in H\}\tilde{p}(Y|\theta, M)d\pi_{\theta|\phi_M,M}(\theta)\right] d\pi_{\phi_M|M}(\phi_M) \\
&= \int_{\Phi_M} \left[\int_\Theta 1\{\theta \in H\}d\pi_{\theta|\phi_M,M}(\theta)\right] p(Y|\phi_M, M)d\pi_{\phi_M|M}(\phi_M) \\
&= \int_{\Phi_M} \pi_{\theta|\phi_M,M}(H)p(Y|\phi_M, M)d\pi_{\phi_M|M}(\phi_M).
\end{aligned}$$

The class of posteriors for θ can be therefore represented as

$$\Pi_{\theta|Y} \equiv \left\{ \sum_{M^p\in\mathcal{M}^p} \pi_{\theta|M^p,Y}\pi_{M^p|Y} + \sum_{M^s\in\mathcal{M}^s} \pi_{\theta|M^s,Y}\pi_{M^s|Y} : \pi_{\theta|M^s,Y} \in \Pi_{\theta|M^s,Y},\ \forall M^s \in \mathcal{M}^s \right\},$$

where $\Pi_{\theta|M^s,Y}$ is as defined in (A.5). As shown in the proof of Lemma A.1 above, marginalizing $\Pi_{\theta|M^s,Y}$ to α leads to $\Pi_{\alpha|M^s,Y}$ defined in (3.7). We therefore conclude that marginalizing $\Pi_{\theta|Y}$ to α results in $\Pi_{\alpha|Y}$ shown in (3.8). ■

Proof of Proposition 3.2. (i) Since there is no constraint across the posteriors belonging to different posterior classes, it holds that

$$\inf_{\pi_{\alpha|Y}\in\Pi_{\alpha|Y}} E_{\alpha|Y}(\alpha) = \sum_{M^p\in\mathcal{M}^p} E_{\alpha|M^p,Y}(\alpha)\pi_{M^p|Y} + \sum_{M^s\in\mathcal{M}^s} \inf_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}} \left\{E_{\alpha|M^s,Y}(\alpha)\right\} \cdot \pi_{M^s|Y}.$$

By the construction of $\Pi_{\alpha|M^s,Y}$, an application of Theorem 2 of Giacomini and Kitagawa (in press) shows the set of posterior means is convex with the lower bound $\inf_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}}\left\{E_{\alpha|M^s,Y}(\alpha)\right\} = E_{\phi_{M^s}|M^s,Y}(l(\phi_{M^s}|M^s))$, and the upper bound $\sup_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}}\left\{E_{\alpha|M^s,Y}(\alpha)\right\} = E_{\phi_{M^s}|M^s,Y}(u(\phi_{M^s}|M^s))$.
(ii) Note that

$$\inf_{\pi_{\alpha|Y}\in\Pi_{\alpha|Y}} \pi_{\alpha|Y}(H) = \sum_{M^p\in\mathcal{M}^p} \pi_{\alpha|M^p,Y}(H)\cdot\pi_{M^p|Y} + \sum_{M^s\in\mathcal{M}^s} \inf_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}} \left\{\pi_{\alpha|M^s,Y}(H)\right\}\cdot\pi_{M^s|Y},$$
$$\sup_{\pi_{\alpha|Y}\in\Pi_{\alpha|Y}} \pi_{\alpha|Y}(H) = \sum_{M^p\in\mathcal{M}^p} \pi_{\alpha|M^p,Y}(H)\cdot\pi_{M^p|Y} + \sum_{M^s\in\mathcal{M}^s} \sup_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}} \left\{\pi_{\alpha|M^s,Y}(H)\right\}\cdot\pi_{M^s|Y}.$$

Theorem 1 of Giacomini and Kitagawa (in press) shows

$$\inf_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}} \left\{\pi_{\alpha|M^s,Y}(H)\right\} = \pi_{\phi_{M^s}|M^s,Y}(IS_\alpha(\phi_{M^s}|M^s)\subset H),$$
$$\sup_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}} \left\{\pi_{\alpha|M^s,Y}(H)\right\} = \pi_{\phi_{M^s}|M^s,Y}(IS_\alpha(\phi_{M^s}|M^s)\cap H\neq\emptyset),$$

so the conclusion follows.
(iii) By setting H to $[-\infty,a]$, the lower probability obtained in part (ii) yields the lower bound of the cdfs, since the event $IS_\alpha(\phi_{M^s}|M^s)\subset[-\infty,a]$ is equivalent to $u(\phi_{M^s}|M^s)\leq a$. The upper bound follows by noting

$$\begin{aligned}\sup_{\pi_{\alpha|M^s,Y}\in\Pi_{\alpha|M^s,Y}} \pi_{\alpha|M^s,Y}([\infty,a]) &= \pi_{\phi_{M^s}|M^s,Y}(IS_\alpha(\phi_{M^s}|M^s)\cap[\infty,a]\neq\emptyset)\\ &= \pi_{\phi_{M^s}|M^s,Y}(l(\phi_{M^s}|M^s)\leq a).\end{aligned}$$

The set of quantiles then follows by inverting these cdf bounds. ■

Next, we show two lemmas to be used to prove Proposition 3.3. We denote the set of candidate models satisfying condition (A) of Assumption 3.2 (i) by $\mathcal{M}_A$ and the set of those satisfying condition (B) by $\mathcal{M}_B$. Under Assumption 3.2 (i), $\mathcal{M}=\mathcal{M}_A\cup\mathcal{M}_B$ holds. Note that through these lemmas and the proof of Proposition 3.3, $\mathcal{M}$ is assumed to admit an identical reduced-form with reduced-form parameter dimension $d\geq 1$.

Lemma A.2 *Suppose Assumption 3.2 holds. For $M\in\mathcal{M}_A$,*

$$\frac{n^{d/2}\det(H_n(\hat{\phi}))^{1/2}p(Y^n|M)}{(2\pi)^{d/2}p(Y^n|\hat{\phi})} - f_{\phi|M}(\hat{\phi}) = O(n^{-1/2}),$$

with $P_{Y^\infty|\phi_{true}}$-probability one.

Proof of Lemma A.2. Denote the reduced-form parameter vector by $\phi=(\phi_1,\ldots,\phi_d)$ and the third-derivative of $l_n(\cdot)$ by $h_{ijk}(\cdot)\equiv\frac{\partial^3}{\partial\phi_i\partial\phi_j\partial\phi_k}l_n(\cdot)$, $1\leq i,j,k\leq d$. By Assumptions 3.2 (i),

(ii) and (iv), there exists B^* an open neighborhood of ϕ_{true} such that $B^* \subset \Phi_M$ holds for all $M \in \mathcal{M}_A$, and

$$\sup_{\phi \in B^*} \max_{1 \le i,j,k \le d} |h_{ijk}(\phi)| < \infty, \tag{A.6}$$

and

$$\limsup_{n \to \infty} \sup_{\phi \in \Phi \setminus B^*} \{l_n(\phi) - l_n(\phi_{true})\} < 0, \quad \text{with } P_{Y^\infty|\phi_{true}}\text{-probability one} \tag{A.7}$$

hold. Since Assumptions 3.2 (iii) and (iv) imply the strong convergence of $\hat{\phi}$, for all sufficiently large n, $\hat{\phi} \in B^*$ holds. Given $\hat{\phi} \in B^*$, consider the third-order mean value expansions of $nl_n(\phi)$:

$$\begin{aligned} nl_n(\phi) &= nl_n(\hat{\phi}) - \frac{n}{2}(\phi - \hat{\phi})' H_n(\hat{\phi})(\phi - \hat{\phi}) + \frac{n}{6} \sum_{1 \le i,j,k \le d} h_{ijk}(\tilde{\phi})(\phi_i - \hat{\phi}_i)(\phi_j - \hat{\phi}_j)(\phi_k - \hat{\phi}_k) \\ &= nl_n(\hat{\phi}) - \frac{1}{2} u' H_n(\hat{\phi}) u + \frac{1}{\sqrt{n}} R_{1n}(u), \end{aligned}$$

where $\tilde{\phi}$ is a convex combination of ϕ and $\hat{\phi}$, $u \equiv \sqrt{n}(\phi - \hat{\phi})$, and $R_{1n}(u) = \frac{1}{6} \sum_{1 \le i,j,k \le d} h_{ijk}(\tilde{\phi}) u_i u_j u_k$, where u_i is the i-th entry of vector u. By the boundedness of h_{ijk} on B^*, $R_{1n}(u)$ can be bounded by a third-order polynomial of u with bounded coefficients on $\sqrt{n}(B^* - \hat{\phi})$, where $\sqrt{n}(B^* - \hat{\phi})$ is the subset in $\mathbb{R}^d$ that translates B^* by $\hat{\phi}$ and scales up by $\sqrt{n}$. Plugging this expansion in $p(Y^n|\phi) = \exp(nl_n(\phi))$ and combining it with the first-order expansion of $f_{\phi|M}(\phi)$, we obtain on $\phi \in B^*$ (or equivalently on $u \in \sqrt{n}(B^* - \hat{\phi})$)

$$\begin{aligned} p(Y^n|\phi) f_{\phi|M}(\phi) &= \exp\left\{ nl_n(\hat{\phi}) - \frac{1}{2} u' H_n(\hat{\phi}) u \right\} \left\{ 1 + \frac{1}{\sqrt{n}} R_{1n}(u) + \frac{1}{2n} R_{1n}(u)^2 + \cdots \right\} \\ &\quad \times \left\{ f_{\phi|M}(\hat{\phi}) + \frac{1}{\sqrt{n}} R_{2n}(u) \right\} \\ &= \exp\left\{ nl_n(\hat{\phi}) - \frac{1}{2} u' H_n(\hat{\phi}) u \right\} \left\{ f_{\phi|M}(\hat{\phi}) + \frac{1}{\sqrt{n}} R_{3n}(u) \right\}, \end{aligned} \tag{A.8}$$

where the first equality invokes the expansion of $\exp(x) = 1 + x + 2^{-1}x^2 + \cdots$, $R_{2n} = f'_{\phi|M}(\tilde{\phi})u$, and R_{3n} collects the residual terms that can be bounded uniformly on $\sqrt{n}(B^* - \hat{\phi})$ by a finite order polynomial of u with bounded coefficients.

Integration of $p(Y^n|\phi) f_{\phi|M}(\phi)$ over $\phi \in B^*$ is equivalent to integrating (A.8) in u over $\sqrt{n}(B^* - \hat{\phi})$:

$$\begin{aligned} &\int_{B^*} p(Y^n|\phi) f_{\phi|M}(\phi) d\phi \\ =& n^{-d/2} \exp\{nl_n(\hat{\phi})\} \left(\int_{\sqrt{n}(B^* - \phi_{true})} \left(f_{\phi|M}(\hat{\phi}) + R_{3n}(u) \right) \exp\left\{ -\frac{1}{2} u' H_n(\hat{\phi}) u \right\} du \right) \\ =& (2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2} \left(f_{\phi|M}(\hat{\phi}) E_{H_n}[1_{\sqrt{n}(B^* - \hat{\phi})}(u)] + n^{-1/2} E_{H_n}[R_{3n}(u) \cdot 1_{\sqrt{n}(B^* - \hat{\phi})}(u)] \right) \\ =& (2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2} \left(f_{\phi|M}(\hat{\phi}) + O(n^{-1/2}) \right), \end{aligned} \tag{A.9}$$

where $E_{H_n}(\cdot)$ is the expectation taken with respect to $u \sim \mathcal{N}(0, H_n(\hat{\phi})^{-1})$. Note that the third equality follows since the replacement of $\sqrt{n}(B^* - \hat{\phi})$ with $\mathbb{R}^d$ incurs an error of exponentially decreasing order and $E_{H_n}(R_{3n}(u))$ is finite, i.e., the multivariate normal distribution has finite moments at any order.

Consider now integrating $p(Y^n|\phi) f_{\phi|M}(\phi)$ over $\Phi_M \setminus B^*$.

$$\begin{aligned}
&\int_{\Phi_M \setminus B^*} p(Y^n|\phi) f_{\phi|M}(\phi) d\phi \\
=&(2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2} \\
&\times \left((2\pi)^{-d/2} n^{d/2} \det(H_n(\hat{\phi}))^{-1/2} \int_{\Phi_M \setminus B^*} \exp\{n(l_n(\phi) - l_n(\hat{\phi}))\} f_{\phi|M}(\phi) d\phi \right) \\
\leq&(2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2} \\
&\times \left((2\pi)^{-d/2} n^{d/2} \det(H_n(\hat{\phi}))^{-1/2} \bar{f}_{\phi|M} \sup_{\phi \in \Phi \setminus B^*} \{\exp\{n(l_n(\phi) - l_n(\phi_{true}))\}\} \right), \qquad \text{(A.10)}
\end{aligned}$$

where by Assumption 3.2 (v), $\bar{f}_{\phi|M} \equiv \sup_{\phi \in \Phi} f_{\phi|M}(\phi) < \infty$. Assumptions 3.2 (iii) and (iv) imply that the term in the parentheses of (A.10) converges to zero faster than $n^{-1/2}$-rate with $P_{Y^\infty|\phi_{true}}$-probability one. Summing up (A.9) and (A.10) gives the following asymptotic approximation of the marginal likelihood in model $M \in \mathcal{M}_A$.

$$\begin{aligned}
p(Y^n|M) &= \int_{B^*} p(Y^n|\phi) f_{\phi|M}(\phi) d\phi + \int_{\Phi_M \setminus B^*} p(Y^n|\phi) f_{\phi|M}(\phi) d\phi \\
&= (2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2} \left(f_{\phi|M}(\hat{\phi}) + O(n^{-1/2}) \right), \qquad \text{(A.11)}
\end{aligned}$$

with $P_{Y^\infty|\phi_{true}}$-probability one. Bringing the multiplicative terms in the right-hand side of (A.11) to the left-hand side completes the proof. ■

Lemma A.3 *Suppose Assumption 3.2 holds. For model $M \in \mathcal{M}_B$,*

$$\frac{n^{d/2} \det(H_n(\hat{\phi}))^{1/2} p(Y^n|M)}{(2\pi)^{d/2} p(Y^n|\hat{\phi})} = o(n^{-1/2}),$$

with $P_{Y^\infty|\phi_{true}}$-probability one.

Proof of Lemma A.3. Let B^* be an open neighborhood of ϕ_{true} as defined in the proof of Lemma A.2.

Consider the marginal likelihood of model $M \in \mathcal{M}_B$ divided by $(2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2}$:

$$\begin{aligned} \frac{n^{d/2}\det(H_n(\hat{\phi}))^{1/2} p(Y^n|M)}{(2\pi)^{d/2} p(Y^n|\hat{\phi})} &= \frac{n^{d/2}\det(H_n(\hat{\phi}))^{1/2}}{(2\pi)^{d/2}} \int_{\Phi_M} \exp\{n(l_n(\phi) - l_n(\hat{\phi}))\} f_{\phi|M}(\phi) d\phi \\ &\leq \frac{n^{d/2}\det(H_n(\hat{\phi}))^{1/2}}{(2\pi)^{d/2}} \bar{f}_{\phi|M} \sup_{\phi \in \Phi_M} \exp\{n(l_n(\phi) - l_n(\hat{\phi}))\} \\ &\leq \frac{n^{d/2}\det(H_n(\hat{\phi}))^{1/2}}{(2\pi)^{d/2}} \bar{f}_{\phi|M} \sup_{\phi \in \Phi \backslash B^*} \exp\{n(l_n(\phi) - l_n(\phi_{true}))\}, \end{aligned} \tag{A.12}$$

where $\bar{f}_{\phi|M} = \sup_{\phi} f_{\phi|M}(\phi) < \infty$, and the third line follows since $B^* \subset \Phi_M^c$ implies $\Phi_M \subset \Phi \backslash B^*$. Note that by Assumption 3.2 (iv), the upper bound shown in (A.12) converges to zero faster than the polynomial rate of $n^{-1/2}$ with $P_{Y^\infty|\phi_{true}}$-probability one. ■

Proof of Proposition 3.3. (i) Under Assumption 3.2 (i), the posterior model probability of model $M \in \mathcal{M}$ can be written as

$$\pi_{M|Y^n} = \frac{p(Y^n|M)\pi_M}{\sum_{M' \in \mathcal{M}_A} p(Y^n|M')\pi_{M'} + \sum_{M' \in \mathcal{M}_B} p(Y^n|M')\pi_{M'}}$$

By dividing both the numerator and denominator by $(2\pi)^{d/2} p(Y^n|\hat{\phi}) n^{-d/2} \det(H_n(\hat{\phi}))^{1/2}$ and applying Lemmas A.2 and A.3, we have

$$\pi_{M|Y^n} = \begin{cases} \frac{f_{\phi|M}(\hat{\phi})\pi_M}{\sum_{M' \in \mathcal{M}_A} f_{\phi|M'}(\hat{\phi})\pi_{M'}} + O(n^{-1/2}), & \text{for } M \in \mathcal{M}_A, \\ o(n^{-1/2}), & \text{for } M \in \mathcal{M}_B, \end{cases}$$

with $P_{Y^\infty|\phi_{true}}$-probability one.

Since $f_{\phi|M}(\cdot)$ is assumed to be continuous and Assumptions 3.2 (iii) and (iv) imply almost sure convergence of $\hat{\phi}$ to ϕ_{true}, $\pi_{M|Y^\infty}$ of the current proposition follows.

(ii) With the given specifications of the ϕ-prior, $f_{\phi|M}(\phi_{true})$ is proportional to $\tilde{\pi}(\Phi_M)^{-1}$ up to the model-independent constant (the Lebesgue density of $\tilde{\pi}_\phi$ evaluated at $\phi = \phi_{true}$). Hence, (i) of the current proposition is reduced to the asymptotic model probabilities of (ii).

(iii) This trivially follows from Lemma 3.2 (iii). ■

A.2 Computing Plausibility Ratios for Sign-restricted SVARs

This appendix provides details on how to compute the posterior-prior plausibility ratios O_M for SVAR models subject to under-identifying zero restrictions and sign restriction. The crucial step is to check if the identified set $IS_\alpha(\phi)$ is empty at ϕ drawn from $\tilde{\pi}_{\phi|Y}$. The first proposal (Algorithm A.1), which is a special case of Algorithm 1 in Giacomini and Kitagawa (in press), uses random draws of the impulse responses and assesses whether any of these satisfies the sign restrictions. The second proposal (Algorithm A.2) directly checks a necessary and sufficient condition for non-emptiness of the identified set. The first algorithm is simple to implement but can give a wrong conclusion if the identified set is tiny. The second algorithm is guaranteed to give the right answer, but can become cumbersome if the number of sign restrictions is large.

A.2.1 Notation

We generalize the representations of the SVAR in (4.1) and the reduced-form VAR in (4.3) to have n endogenous variables and $p \geq 0$ lags. Let $Q \in \mathcal{O}(n)$ be an $n \times n$ orthonormal matrix and $\mathcal{O}(n)$ be the set of $n \times n$ orthonormal matrices. We first transform the structural parameters $(A_0, a, A_1, \ldots, A_p)$ into $(\phi', vec(Q)')' \in \tilde{\Phi} \times vec(\mathcal{O}(n))$:

$$\begin{aligned}
&B = A_0^{-1}\left[a, A_1, \ldots, A_p\right],\\
&\Sigma = A_0^{-1}\left(A_0^{-1}\right)',\\
&Q = \Sigma_{tr}^{-1} A_0^{-1},
\end{aligned}$$

where Σ_{tr} denotes the lower-triangular Cholesky factor of Σ with nonnegative diagonal elements. We then set $\theta = (\phi', vec(Q)')'$ with domain $\Theta = \left\{(\phi', vec(Q)')' \in \Phi \times vec(\mathcal{O}(n)) : diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0\right\}$. Here, $diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0$ is the sign normalization restrictions:

$$\left(\sigma^i\right)' q_i \geq 0 \quad \text{for all } i = 1, \ldots, n, \tag{A.13}$$

where $\left[\sigma^1, \sigma^2, \ldots, \sigma^n\right]$ are the columns of Σ_{tr}^{-1} and $[q_1, q_2, \ldots, q_n]$ are the columns of Q.

Assuming the lag polynomial $\left(I_n - \sum_{j=1}^{p} B_j L^p\right)$ is invertible (which is the domain restriction on $\tilde{\Phi}$) the VMA(∞) representation of the model is:

$$\begin{aligned}
y_t &= c + \sum_{j=0}^{\infty} C_j u_{t-j} \qquad \text{(A.14)}\\
&= c + \sum_{j=0}^{\infty} C_j \Sigma_{tr} Q \epsilon_{t-j},
\end{aligned}$$

where C_j is the j-th coefficient matrix of $\left(I_n - \sum_{j=1}^{p} B_j L^j\right)^{-1}$.

We denote the h-th horizon impulse response by the $n \times n$ matrix IR^h, $h = 0, 1, 2, \ldots$

$$IR^h = C_h \Sigma_{tr} Q. \tag{A.15}$$

The scalar parameter of interest α is a single impulse-response, i.e., the (i, j)-element of IR^h, which can be expressed as

$$\alpha = IR_{ij}^h \equiv e_i' C_h \Sigma_{tr} Q e_j \equiv c_{ih}'(\phi)\, q_j, \tag{A.16}$$

where e_i is the i-th column of the identity matrix I_n and $c'_{ih}(\phi)$ is the i-th row of $C_h \Sigma_{tr}$.

Zero restrictions can be imposed on off-diagonal elements of A_0, lagged coefficients $\{A_l : l = 1, \ldots p\}$, contemporaneous impulse responses $IR^0 = A_0^{-1}$, or cumulative long-run responses. All these restrictions can be viewed as linear constraints on the columns of Q. For example:

$$\begin{aligned}
((j,i)\text{-th element of } A_0) &= 0 \iff \left(\Sigma_{tr}^{-1} e_i\right)' q_j = 0, \\
((j,i)\text{-th element of } A_l) &= 0 \iff \left(\Sigma_{tr}^{-1} B_l e_i\right)' q_j = 0, \\
\left((i,j)\text{-th element of } A_0^{-1}\right) &= 0 \iff \left(e_i' \Sigma_{tr}\right) q_j = 0, \\
\left((i,j)\text{-th element of } IR^h\right) &= 0 \iff \left[e_i' C_h \Sigma_{tr}\right] q_j = 0.
\end{aligned} \tag{A.17}$$

We restrict our analysis to the case that the imposed zero restrictions constrain only one column vector of Q. Ordering the variables in such way that q_1 becomes the constrained column vector of Q, we can represent a collection of zero restrictions as

$$F(\phi) q_1 = \mathbf{0}, \tag{A.18}$$

where $F(\phi)$ is an $f \times n$ matrix. $F(\phi)$ stacks all the coefficient vectors that multiply q_1 into a matrix. Hence, f is the number of imposed zero restrictions. We consider under-identifying zero restrictions, so we assume $f \leq n - 2$.

We suppose there are sign restrictions on the responses to the first structural shock. Sign restrictions are linear constraints on the first column of Q: $S_h(\phi) q_1 \geq \mathbf{0}$, where $S_h(\phi) \equiv D_h C_h \Sigma_{tr}$ is an $s_h \times n$ matrix, and D_h is an $s_h \times n$ matrix that selects the sign-restricted responses from the $n \times 1$ impulse-response vector $C_h \Sigma_{tr} q_1$. The nonzero elements of D_h equal 1 or -1 depending on whether the corresponding impulse responses are positive or negative.

Stacking $S_h(\phi)$ over multiple horizons gives the set of sign restrictions

$$S(\phi) q_1 \geq \mathbf{0}, \tag{A.19}$$

where $S(\phi)$ is a $s \times n$ matrix $S(\phi) = \left[S_0(\phi)', \ldots, S_{\bar{h}}(\phi)\right]'$, where $s = \sum_{h=0}^{\bar{h}} s_h$ is the number of sign constraints and $0 \leq \bar{h} \leq \infty$ is the maximal horizon in the impulse-response analysis.[1]

A.2.2 Algorithms

For multiple posterior models, the plausibility ratio O_M can be computed by plugging into (3.5) numerical approximations of the prior and posterior probabilities for non-emptiness of the identified set. Specifically, the denominator of O_M can be computed by drawing many ϕ's from the prior and finding the fraction of draws that yield a non-empty identified set. The numerator of O_M can be computed similarly except that the ϕ's are drawn from the posterior.

[1] If there are no sign restrictions on the $\tilde{h}$-th horizon responses, $\tilde{h} \in \{0, \ldots, \bar{h}\}$, $s_{\tilde{h}} = 0$ and $S_{\tilde{h}}(\phi)$ is not present in $S(\phi)$.

Our first algorithm to approximate O_M draws many q_1's from a distribution supported only on the unit sphere, and checks if any of the draws satisfies the model's assumptions given ϕ.

Algorithm A.1 *Suppose the identifying restrictions of model M consist of $F(\phi)$ and $S(\phi)$ be the zero and sign restrictions as defined in (A.18) and (A.19), respectively. The following algorithm can be used to approximate $\tilde{p}_\phi(\Phi_M)$, where $\tilde{p}_\phi$ is a probability measure on $\tilde{\Phi}$, which can be $\tilde{\pi}_\phi$ or $\tilde{\pi}_{\phi|Y}$.*

1. *Draw ϕ from $\tilde{p}_\phi$.*
2. *Let $z \sim \mathcal{N}(\mathbf{0}, I_n)$ be a draw of an n-variate standard normal random variable. Let $\tilde{q}_1 = \mathcal{M}z$ be the $n \times 1$ residual vector in the linear projection of z onto the $n \times f$ regressor matrix $F(\phi)'$. Set $q_1 = sign\left(\left(\sigma^1\right)' \tilde{q}_1\right) \frac{\tilde{q}_1}{\|\tilde{q}_1\|}$. If $\left(\sigma^i\right)' \tilde{q}_i$ is zero for some i, set $sign\left(\left(\sigma^i\right)' \tilde{q}_i\right)$ equal to 1 or -1 with equal probability.*
3. *Check if q_1 satisfies the sign restrictions $S(\phi)q_1 \geq 0$. If it does, we conclude $IS_\alpha(\phi) \neq \emptyset$. Otherwise, repeat Step 2 a maximum of L times until q_1 satisfying $S(\phi)q_1 \geq \mathbf{0}$ is obtained. If none of the L draws of q_1 satisfies $S(\phi, Q) \geq \mathbf{0}$, approximate $IS_\alpha(\phi)$ as being empty and return to Step 1 to obtain a new draw of ϕ.*
4. *Repeat Steps 1 – 3 for K times. The proportion of drawn ϕ's that gives non-empty $IS_\alpha(\phi)$ in Step 3 approximates $\tilde{p}_\phi(\Phi_M)$.*

This procedure is simple to implement and can be applied where the number of sign restrictions is large. It however only delivers an approximate assessment of the identified set non-emptiness, and the approximation can be poor if the set of q's satisfying the sign restrictions is so thin that the finite number of q_1 draws misses it.

The next algorithm exploits the linear structure of the identifying restrictions and does not rely on approximations. The algorithm is based on the observation that any non-empty identified set for q_1 contains a vertex on the unit sphere on which at least $n-1$ equality and inequality constraints are binding. We can exhaust all the possible candidates for such vertex by selecting any combination of $n-1$ constraints and setting them binding. If we could find a vertex that satisfies the $f + s - (n-1)$ constraints ruled out in the selection, we can claim this vertex is contained in the identified set for q_1, allowing us to conclude that it is non-empty. If we cannot find any such vertex, we conclude that the identified set is empty.

Algorithm A.2 *Suppose the identifying restrictions of model M consist of $F(\phi)$ and $S(\phi)$ be the zero and sign restrictions as defined in (A.18) and (A.19), respectively. The following algorithm can be used to approximate $\tilde{p}_\phi(\Phi_M)$, where $\tilde{p}_\phi$ is a probability measure on $\tilde{\Phi}$, which can be $\tilde{\pi}_\phi$ or $\tilde{\pi}_{\phi|Y}$.*

1. *Draw ϕ from $\tilde{p}_\phi$.*

2. *Find unit length vectors q_1^* and $-q_1^*$ satisfying the system of "active constraints" (in the language of Gafarov et al. (2018)):*

$$\begin{cases} F(\phi)q = 0 \\ \tilde{S}(\phi)q = 0 \end{cases} \tag{A.20}$$

where $\tilde{S}(\phi)$ is $\tilde{s} \times n$ matrix of active sign restrictions. It is set by picking $\tilde{s}$ rows from $S(\phi)$ matrix, where $f + \tilde{s} = n - 1$. Check if q_1^* or $-q_1^*$ satisfy the "inactive constraints," namely the rest of sign restrictions and the sign normalization restriction for q_1. If so, $IS_\alpha(\phi)$ is non-empty. Otherwise, keep constructing $\tilde{S}(\phi)$ with different combinations of $\tilde{s}$ active constraints and verify if the corresponding solution satisfy the inactive constraints. If none of the solutions satisfies the inactive restrictions, $IS_\alpha(\phi)$ is empty.

3. *Repeat Step 1 – 2 K times.*

4. *Approximate $\tilde{p}_\phi(\Phi_M)$ by the proportion of K draws of ϕ that delivers non-empty identified set in Step 2.*

While this algorithm does not suffer from approximation error, it can become computationally burdensome when there are many sign restrictions, as the number of combinations of the active constraints to be checked in Step 2 becomes very large. The algorithm of Amir-Ahmadi and Drautzburg (2020) (Section 3.2) works without approximation error and can be applied to the current context. The main difference is that they solve a constrained optimization problem to detect non-emptiness. They check that the Chebychev center of the constrained set (prior to normalization) is nondegenerate: the existence of a Chebychev center with a ball of radius strictly positive around it is equivalent to an identified set with positive measure.

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