A Generalized Approach to Indeterminacy in Linear Rational Expectations Models

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1 Online Appendix: Bivariate analytic example

In this online Appendix, we provide an analytical example to show the following results: First, the equivalence between the solutions for an indeterminate LRE model using the methodology of Lubik and Schorfheide (2004) and our proposed method. Second, there exists a unique mapping between the alternative representations that can be considered using our augmented representation: The alternative representations are equivalent up to a transformation of the correlations between the exogenous shocks and the forecast error included in the auxiliary process.

1.1 Lubik and Schorfheide (2004)

We consider the following simple model

$$y_t = \frac{1}{\theta_y} E_t(y_{t+1}) + \frac{1}{\theta_y} E_t(x_{t+1}) + \varepsilon_t \quad (1)$$

$$x_t = \frac{1}{\theta_x} E_t(x_{t+1}) \quad (2)$$

where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$ and the corresponding forecast errors are denoted as

$$\eta_{y,t} \equiv y_t - E_{t-1}(y_t) \quad (3)$$

$$\eta_{x,t} \equiv x_t - E_{t-1}(x_t) \quad (4)$$

The LRE model in (1)$\sim$ (4) can be written in the following matrix form

$$\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi\varepsilon_t + \Pi\eta_t, \tag{5}$$

where $S_t \equiv (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}))'$ and $\eta_t \equiv (\eta_{y,t}, \eta_{x,t})'$.
As the matrix Γ_0 is non-singular, the LRE model in (5) can be written as

$$S_t = \Gamma_1^* S_{t-1} + \Psi^*\varepsilon_t + \Pi^*\eta_t, \tag{6}$$

where

$$\Gamma_1^* \equiv \Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} \mathbf{0_{4\times 2}} & \mathbf{A_{4\times 2}} \end{bmatrix}, \qquad \Pi^* \equiv \Gamma_0^{-1}\Pi = \mathbf{A_{4\times 2}}$$

$$\Psi^* \equiv \Gamma_0^{-1}\Psi = \begin{bmatrix} 0 \\ 0 \\ -\theta_x \\ 0 \end{bmatrix}, \qquad \mathbf{A_{4\times 2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \theta_y & -\theta_x \\ 0 & \theta_x \end{bmatrix}$$

Applying the Jordan decomposition, the matrix Γ_1^* can be decomposed as $\Gamma_1^* \equiv J\Lambda J^{-1}$, where the elements of the diagonal matrix Λ denote the roots of the system

$$\Lambda \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_x & 0 \\ 0 & 0 & 0 & \theta_y \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \mathbf{0} \\ \mathbf{0} & \theta_y \end{bmatrix}.$$

Assuming without loss of generality that $|\theta_x| \leq 1$ and $|\theta_y| > 1$, the system in (6) is indeterminate because the number of expectational variables, $\{E_t(y_{t+1}), E_t(x_{t+1})\}$, exceeds the number of explosive roots, θ_y. Defining the vector $w_t \equiv J^{-1}S_t$, the model can be represented as

$$w_t \equiv \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \mathbf{0} \\ \mathbf{0} & \theta_y \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix} \eta_t, \tag{7}$$

where the first block denotes the stationary block of the system and the second block is unstable.

The adoption of Sims' (2002) code, Gensys, to solve this model is not appropriate as it deals with determinate models. After having obtained the representation in (7), Gensys would construct a matrix Φ such that premultiplying the system by a matrix $[I \;\; -\Phi]$ would eliminate the effect of non-fundamental shocks. Equivalently, the matrix has to satisfy the condition

$$[I \quad -\Phi]\begin{bmatrix}\tilde{\Pi}_1\\ \tilde{\Pi}_2\end{bmatrix} = \tilde{\Pi}_1 - \Phi\tilde{\Pi}_2 = 0. \tag{8}$$

Under determinacy, the matrix $\tilde{\Pi}_2$ is square and, assuming that it is also non-singular[1], it is possible to solve for $\Phi = \tilde{\Pi}_1\left(\tilde{\Pi}_2\right)^{-1}$.

The approach in Lubik and Schorfheide (2004) modifies this intuition to account for the indeterminacy that characterizes the model in (7). Under indeterminacy, the matrix $\tilde{\Pi}_2$ is a vector with more columns than rows, implying that it is not possible to obtain a matrix Φ that satisfies the above condition in (8). Nevertheless, Lubik and Schorfheide (2004) apply a singular value decomposition (SVD) to the matrix $\tilde{\Pi}_2$ to obtain

$$\tilde{\Pi}_2 \equiv UDV' = \begin{bmatrix}U_{.1} & U_{.2}\end{bmatrix}\begin{bmatrix}D_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V'_{.1}\\ V'_{.2}\end{bmatrix} = U_{.1}D_{11}V'_{.1}, \tag{9}$$

where D_{11} is a diagonal matrix and U and V are orthonormal matrices. In this particular example, the matrix to decompose is $\tilde{\Pi}_2 = \begin{bmatrix}a & b\end{bmatrix}$, where $a \equiv -\theta_y$ and $b \equiv -\theta_x\theta_y/(\theta_x - \theta_y)$, and the resulting SVD is

$$\tilde{\Pi}_2 \equiv UDV' = 1\begin{bmatrix}d & 0\end{bmatrix}\begin{bmatrix}\frac{a}{d} & \frac{b}{d}\\ \frac{b}{d} & -\frac{a}{d}\end{bmatrix}, \tag{10}$$

where $d \equiv \sqrt{a^2+b^2}$. Lubik and Schorfheide (2004) then proceed by defining the matrix Φ as

$$\Phi = \tilde{\Pi}_1\left(V_{.1}d^{-1}U'_{.1}\right) = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & \theta_x\end{bmatrix}\begin{bmatrix}\frac{a}{d}\\ \frac{b}{d}\end{bmatrix}\frac{1}{d} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & \theta_x\frac{b}{d^2}\end{bmatrix},$$

and premultiply the system in (7) by the following matrices

$$\begin{aligned}\begin{bmatrix}I & -\Phi\\ 0 & 1\end{bmatrix}\begin{bmatrix}w_{1,t}\\ w_{2,t}\end{bmatrix} \quad = \quad & \begin{bmatrix}I & -\Phi\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Lambda_{11} & 0\\ 0 & \theta_y\end{bmatrix}\begin{bmatrix}w_{1,t-1}\\ w_{2,t-1}\end{bmatrix} + \\ & + \begin{bmatrix}I & -\Phi\\ 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{\Psi}_1\\ \tilde{\Psi}_2\end{bmatrix}\varepsilon_t + \underbrace{\begin{bmatrix}I & -\Phi\\ 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{\Pi}_1\\ \tilde{\Pi}_2\end{bmatrix}}_{\neq 0}\eta_t,\end{aligned} \tag{11}$$

[1]Note that Gensys obtains the matrix Φ even when the matrix $\tilde{\Pi}_2$ is singular by applying a singular value decomposition.

where the second block represents the constraint that guarantees the boundedness of the solution,

$$w_{2,t} = 0 \Longleftrightarrow E_t(y_{t+1}) = -\frac{b}{a} E_t(x_{t+1}). \tag{12}$$

Importantly, given that the model is indeterminate, the last term in equation (11) differs from zero and therefore non-fundamental disturbances affect the model dynamics. Solving (11) for the endogenous variables, S_t, the system takes the form

$$S_t = \tilde{\Gamma}_1^* S_{t-1} + \tilde{\Psi}^* \varepsilon_t + \tilde{\Pi}^* \eta_t, \tag{13}$$

where

$$\tilde{\Gamma}_1^* \equiv \begin{bmatrix} \mathbf{0_{4\times 2}} & \mathbf{B_{4\times 2}} \end{bmatrix}, \qquad \tilde{\Psi}^* \equiv \left(\frac{a}{d}\right)^2 \begin{bmatrix} 1 \\ b/a \\ -\theta_x (b/a)^2 \\ \theta_x b/a \end{bmatrix},$$

$$\tilde{\Pi}^* \equiv \mathbf{B_{4\times 2}} = \begin{bmatrix} \left(b^2/d^2\right) & -\frac{b}{a}(1 - b^2/d^2) \\ -ab/d^2 & (1 - b^2/d^2) \\ \theta_x \left(b^2/d^2\right) & -\theta_x \frac{b}{a}(1 - b^2/d^2) \\ -\theta_x ab/d^2 & \theta_x (1 - b^2/d^2) \end{bmatrix}.$$

The last step that Lubik and Schorfheide (2004) implement is to express the forecast errors as a function of the fundamental shock, ε_t, and a sunspot shock, ζ_t, as

$$\eta_t = -V_{.1} D_{11}^{-1} U_{.1}' \tilde{\Psi}_2 \varepsilon_t + V_{.2} \left(\breve{M} \varepsilon_t + M_\zeta \zeta_t \right), \tag{14}$$

where $V_{.2}' = \begin{bmatrix} \frac{b}{d} & -\frac{a}{d} \end{bmatrix}$. Combining (13) with (14) and normalizing $M_\zeta = 1$, the solution to the LRE model is[2]

$$S_t = \tilde{\Gamma}_1^* S_{t-1} + \tilde{\Psi}^* \varepsilon_t + \tilde{\Pi}^* V_{.2} \left(\breve{M} \varepsilon_t + \zeta_t \right). \tag{15}$$

This solution can be *equivalently* written in a form that explicitly includes the boundedness condition in (12) for which $w_{2,t} = 0$ and therefore $E_t(y_{t+1}) = -\frac{b}{a} E_t(x_{t+1})$. Recalling that $S_t = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}))'$, the dynamics of the solution in (15) are now expressed as a function of only one state variable,

[2]Note that the term $-\tilde{\Pi}^* \left(V_{.1} D_{11}^{-1} U_{.1}' \breve{\Psi}_2 \right) \varepsilon_t$ always equals to zero since $\left(\tilde{\Pi}^* V_{.1} \right) = 0$ by the properties of the orthonormal matrix V.

$$
\begin{aligned}
S_t &= \begin{bmatrix} -b/a \\ 1 \\ -\theta_x b/a \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \tilde{\Psi}^* \varepsilon_t + \tilde{\Pi}^* V_{.2} \left(\breve{M} \varepsilon_t + \zeta_t \right) \\
&= \begin{bmatrix} \frac{\theta_x}{(\theta_y - \theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y - \theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \frac{\theta_y^2}{d^2} \begin{bmatrix} 1 \\ \frac{\theta_x}{(\theta_x - \theta_y)} \\ -\frac{\theta_x^3}{(\theta_x - \theta_y)^2} \\ \frac{\theta_x^2}{(\theta_x - \theta_y)} \end{bmatrix} \varepsilon_t + \frac{\theta_y}{d} \begin{bmatrix} \frac{\theta_x}{(\theta_y - \theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y - \theta_x)} \\ \theta_x \end{bmatrix} \left(\breve{M} \varepsilon_t + \zeta_t \right), \qquad (16)
\end{aligned}
$$

where $d = \sqrt{\theta_y^2 + (\theta_x \theta_y)^2 / (\theta_x - \theta_y)^2}$.

1.2 Our proposed methodology

We now provide the derivation of the solution for the LRE model in (5) and reported below in equation (17) using the methodology proposed in this paper

$$
\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \qquad (17)
$$

The methodology consists of appending the following equation to the original LRE model

$$
\omega_t = \frac{1}{\alpha} \omega_{t-1} + \nu_{x,t} - \eta_{x,t},
$$

where v_t denotes a newly defined sunspot shock and without loss of generality $\alpha \equiv |\theta_x|$. Denoting the newly defined vector of endogenous variables $\hat{S}_t \equiv (S_t, \omega_t)' = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}), \omega_t)'$, and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t^x \equiv (\varepsilon_t, \nu_{x,t})'$, the augmented representation of the LRE model is

$$
\hat{\Gamma}_0 \hat{S}_t = \hat{\Gamma}_1 \hat{S}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t^x + \hat{\Pi} \eta_t. \qquad (18)
$$

Pre-multiplying the system in (18) by $\hat{\Gamma}_0^{-1}$, we obtain

$$
\hat{S}_t = \hat{\Gamma}_1^* \hat{S}_{t-1} + \hat{\Psi}^* \hat{\varepsilon}_t^x + \hat{\Pi}^* \eta_t, \qquad (19)
$$

where

$$\hat{\Gamma}_1^* \equiv \begin{bmatrix} \Gamma_1^* & \mathbf{0_{4\times 1}} \\ & \\ \mathbf{0_{1\times 4}} & \frac{1}{\alpha} \end{bmatrix}, \quad \hat{\Psi}^* \equiv \begin{bmatrix} \Psi^* & \mathbf{0_{4\times 1}} \\ & \\ 0 & -1 \end{bmatrix}, \quad \hat{\Pi}^* \equiv \begin{bmatrix} & \Pi_{4\times 2}^* \\ & \\ 0 & 1 \end{bmatrix}.$$

and the matrices $\{\Gamma_1^*, \Psi^*, \Pi^*\}$ are the same as those found in (6). Applying the Jordan decomposition, the matrix $\hat{\Gamma}_1^*$ can be decomposed as $\hat{\Gamma}_1^* \equiv \hat{J}\hat{\Lambda}\hat{J}^{-1}$, where the elements of the diagonal matrix $\hat{\Lambda}$ denote the roots of the system

$$\hat{\Lambda} \equiv \begin{bmatrix} \Lambda & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_x & 0 & 0 \\ 0 & 0 & 0 & \theta_y & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}.$$

Assuming as in the previous section that $|\theta_x| \leq 1$ and $|\theta_y| > 1$, then $1/\alpha = 1/|\theta_x| > 1$ and the diagonal elements of the matrix $\Lambda_{22} = \begin{bmatrix} \theta_y & 0 \\ 0 & 1/\alpha \end{bmatrix}$ correspond to the explosive roots of the system. While the original system in (17) is indeterminate, the augmented representation in (18) is determinate as the number of expectational variables, $\{E_t(y_{t+1}), E_t(x_{t+1})\}$, equals the number of explosive roots, $\{\theta_y, 1/\alpha\}$. Defining the vector $\hat{w}_t \equiv \hat{J}^{-1}\hat{S}_t$, the model can be represented as

$$\hat{w}_t \equiv \begin{bmatrix} \hat{w}_{1,t} \\ \hat{w}_{2,t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \hat{w}_{1,t-1} \\ \hat{w}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \hat{\Psi}_1^{**} \\ \hat{\Psi}_2^{**} \end{bmatrix} \hat{\varepsilon}_t^x + \begin{bmatrix} \hat{\Pi}_1^{**} \\ \hat{\Pi}_{2,x}^{**} \end{bmatrix} \eta_t, \tag{20}$$

where the first block is stationary. Given that the second block is unstable, the following two conditions have to be imposed to guarantee the boundedness of the solution. First, the linear combination of the endogenous variables, $\hat{w}_{2,t}$, is set to zero,

$$\hat{w}_{2,t} = 0 \iff \begin{cases} E_t(y_{t+1}) = -\frac{b}{a} E_t(x_{t+1}) \\ \omega_t = 0 \end{cases} \tag{21}$$

Second, the linear combination of fundamental and non-fundamental shocks also has to equal zero. Therefore, the non-fundamental shocks, η_t, become a function of the augmented vector of exogenous shocks, $\hat{\varepsilon}_t^x$,

$$\eta_t = -\left(\hat{\Pi}_{2,x}^{**}\right)^{-1} \hat{\Psi}_2^{**} \hat{\varepsilon}_t^x \iff \eta_t = \begin{bmatrix} 1 & -\frac{\theta_x}{\theta_x - \theta_y} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \nu_{x,t} \end{bmatrix} \tag{22}$$

Considering equation (20), it is relevant to point out that the matrix $\hat{\Pi}^{**}_{2,x}$ differs from the corresponding matrix for the representation in which we incorporate the forecast error, $\eta_{y,t}$, defined as $\hat{\Pi}^{**}_{2,y}$,

$$\hat{\Pi}^{**}_{2,x} \equiv \begin{bmatrix} \theta_y & \frac{\theta_x\theta_y}{\theta_x-\theta_y} \\ 0 & -1 \end{bmatrix} \qquad \hat{\Pi}^{**}_{2,y} \equiv \begin{bmatrix} \theta_y & \frac{\theta_x\theta_y}{\theta_x-\theta_y} \\ -1 & 0 \end{bmatrix}.$$

Therefore, when the auxiliary process is written as a function of the non-fundamental shock, $\eta_{y,t}$, the restriction imposed on η_t to guarantee the boundedness of the solution also differs from the one found in (22)

$$\eta_t = -\left(\hat{\Pi}^{**}_{2,y}\right)^{-1}\hat{\Psi}^{**}_2\hat{\varepsilon}^y_t \iff \eta_t = \begin{bmatrix} 0 & 1 \\ \frac{\theta_x-\theta_y}{\theta_x} & -\frac{\theta_x-\theta_y}{\theta_x} \end{bmatrix}\begin{bmatrix} \varepsilon_t \\ \nu_{y,t} \end{bmatrix} \tag{23}$$

Importantly, from equations (22) and (23) it is possible to establish a relationship that links the two non-fundamental disturbances $\{\nu_{x,t}, \nu_{y,t}\}$ and the exogenous shock ε_t,

$$\nu_{x,t} = \frac{\theta_x-\theta_y}{\theta_x}\varepsilon_t - \frac{\theta_x-\theta_y}{\theta_x}\nu_{y,t}. \tag{24}$$

We show below that equations (21) and (24) are crucial for the equivalence between the augmented representations that include different non-fundamental shocks in the auxiliary processes that our methodology proposes.

The augmented model in (20) is determinate as the second block has two explosive roots to match the two expectational variables of the model. It is therefore possible to apply the approach in Sims'(2002) to construct a matrix $\hat{\Phi}_x$ such that premultiplying the system by a matrix $[I \;\; -\hat{\Phi}_x]$ would eliminate the effect of non-fundamental shocks. Equivalently, the matrix has to satisfy the condition

$$[I \;\; -\hat{\Phi}_x]\begin{bmatrix} \hat{\Pi}^{**}_1 \\ \hat{\Pi}^{**}_{2,x} \end{bmatrix} = \hat{\Pi}^{**}_1 - \hat{\Phi}_x\hat{\Pi}^{**}_{2,x} = 0. \tag{25}$$

Importantly, the matrix $\hat{\Pi}^{**}_{2,x}$ is square under determinacy and, assuming that it is also non-singular[3], it is possible to solve for $\hat{\Phi}_x = \hat{\Pi}^{**}_1\left(\hat{\Pi}^{**}_{2,x}\right)^{-1}$.

[3]Note that Gensys obtains the matrix $\hat{\Phi}$ even when the matrix $\hat{\Pi}^{**}_2$ is singular by applying a singular value decomposition.

To solve the model, the system in (20) is then premultiplied by the following matrices

$$\begin{aligned}\begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{w}_{1,t} \\ \hat{w}_{2,t} \end{bmatrix} &= \begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}\begin{bmatrix} \hat{w}_{1,t-1} \\ \hat{w}_{2,t-1} \end{bmatrix} + \\ &+ \begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\Psi}_1^{**} \\ \hat{\Psi}_2^{**} \end{bmatrix}\hat{\varepsilon}_t^x + \underbrace{\begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\Pi}_1^{**} \\ \hat{\Pi}_{2,x}^{**} \end{bmatrix}}_{=0}\eta_t, \end{aligned} \tag{26}$$

where the second block represents the constraint that guarantees the boundedness of the solution, $\hat{w}_{2,t} = 0$. Importantly, the augmented representation is determinate, and the last term of the system in (26) equals zero. Nevertheless, the non-fundamental disturbance, $\nu_{x,t}$, affects the dynamics of the original model through vector of exogenous shocks, $\hat{\varepsilon}_t^x \equiv (\varepsilon_t, \nu_{x,t})'$. Solving (25) for the endogenous variables, the system takes the form

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} \nu_{x,t}. \tag{27}$$

1.3 Equivalence of alternative representations under our proposed method

We now show that there exists a unique mapping between the alternative representations that can be considered using our augmented representation: These representations are equivalent up to a transformation of the correlations between the exogenous shocks and the forecast error included in the auxiliary process. In particular, to rewrite the reduced-form solution for the augmented representation that includes the non-fundamental shock, $\eta_{y,t}$, in the auxiliary process, we recall equations (21) and (24) that we report below in equations (28) and (29)

$$\hat{w}_{2,t} = 0 \iff \begin{cases} E_t(y_{t+1}) = -\frac{\theta_x}{\theta_x-\theta_y} E_t(x_{t+1}) \\ \omega_t = 0 \end{cases} \tag{28}$$

$$\nu_{x,t} = \frac{\theta_x - \theta_y}{\theta_x}\varepsilon_t - \frac{\theta_x - \theta_y}{\theta_x}\nu_{y,t} \tag{29}$$

Using the above equations, we can rewrite the system in (27) as a function of $\nu_{y,t}$, rather than

$\nu_{x,t}$,

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\theta_y-\theta_x}{\theta_x} \\ \theta_x \\ \theta_y-\theta_x \end{bmatrix} E_{t-1}(y_t) + \begin{bmatrix} 0 \\ \frac{\theta_x-\theta_y}{\theta_x} \\ -\theta_x \\ \theta_x-\theta_y \end{bmatrix} \varepsilon_t + \begin{bmatrix} 1 \\ -\frac{\theta_x-\theta_y}{\theta_x} \\ \theta_x \\ -(\theta_x-\theta_y) \end{bmatrix} \nu_{y,t}. \tag{30}$$

1.4 Equivalence of methodologies: Lubik and Schorfheide (2004) and our proposed method

In this section, we show the equivalence of the representations obtained using the two methodologies. In equation (31) below, we report the solution for the endogenous variables, $S_t = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}))'$, using the methodology of Lubik and Schorfheide (2004),

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \frac{\theta_y^2}{d^2} \begin{bmatrix} 1 \\ \frac{\theta_x}{(\theta_x-\theta_y)} \\ -\frac{\theta_x^3}{(\theta_x-\theta_y)^2} \\ \frac{\theta_x^2}{(\theta_x-\theta_y)} \end{bmatrix} \varepsilon_t + \frac{\theta_y}{d} \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} \left(\breve{M}\varepsilon_t + \zeta_t\right), \tag{31}$$

where $d = \sqrt{\theta_y^2 + (\theta_x\theta_y)^2/(\theta_x-\theta_y)^2}$. We now report in equation (32) below the solution using our methodology when we include the forecast error, $\eta_{x,t}$, in the auxiliary process

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} \nu_{x,t}. \tag{32}$$

To show the equivalence between the two representations, we need to recall the restrictions that each methodology imposed on the forecast errors, η_t, as a function of the exogenous shock, ε_t, and the additional sunspot shock. Following Lubik and Schorfheide (2004), we derived that

$$\eta_t = -V_{.1}D_{11}^{-1}U_{.1}'\tilde{\Psi}_2\varepsilon_t + V_{.2}\left(\breve{M}\varepsilon_t + M_\zeta\zeta_t\right),$$

where we know that $V' = \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} = \begin{bmatrix} \frac{a}{d} & \frac{b}{d} \\ \frac{b}{d} & -\frac{a}{d} \end{bmatrix}$, $D_{11} = d = \sqrt{a^2+b^2}$, $U_1 = 1$, $\tilde{\Psi}_2 = -a = \theta_y$ and

$b = -\theta_x\theta_y/(\theta_x - \theta_y)$. Therefore, normalizing $M_\zeta = 1$, we obtain

$$\begin{aligned}\eta_t &= \begin{bmatrix}\frac{a}{d}\\ \frac{b}{d}\end{bmatrix}\frac{a}{d}\varepsilon_t + \begin{bmatrix}\frac{b}{d}\\ -\frac{a}{d}\end{bmatrix}\left(\breve{M}\varepsilon_t + \zeta_t\right)\\ &= \left\{\frac{\theta_y^2}{d^2}\begin{bmatrix}1\\ \frac{\theta_x}{(\theta_x-\theta_y)}\end{bmatrix} + \frac{\theta_y}{d}\begin{bmatrix}-\frac{\theta_x}{(\theta_x-\theta_y)}\\ 1\end{bmatrix}\breve{M}\right\}\varepsilon_t + \frac{\theta_y}{d}\begin{bmatrix}-\frac{\theta_x}{(\theta_x-\theta_y)}\\ 1\end{bmatrix}\zeta_t. \end{aligned} \tag{33}$$

Similarly, from the derivation using our methodology, we know that

$$\eta_t = -\left(\hat{\Pi}^{**}_{2,x}\right)^{-1}\hat{\Psi}^{**}_2\hat{\varepsilon}^x_t \iff \eta_t = \begin{bmatrix}1 & -\frac{\theta_x}{\theta_x-\theta_y}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon_t\\ \nu_{x,t}\end{bmatrix} \tag{34}$$

Comparing equations (33) and (34), we also point out that the sunspot shock introduced in our representation, $\nu_{x,t}$, has a clear interpretation: It is always equivalent to the forecast error that is included in the auxiliary process. On the contrary, the sunspot shock, ζ_t, in Lubik and Schorfheide (2003) has a more complex interpretation and the authors provide a formal argument to consider it as a trigger of belief shocks that lead to a revision of the forecasts.

We then combine equations (33) and (34) to establish the following relationship

$$\nu_{x,t} = \left[\frac{\theta_y^2}{d^2}\frac{\theta_x}{(\theta_x-\theta_y)} + \frac{\theta_y}{d}\breve{M}\right]\varepsilon_t + \frac{\theta_y}{d}\zeta_t. \tag{35}$$

Plugging this relationship in the solution in equation (32) obtained using our methodology, we derive the solution in (31) derived using the methodology of Lubik and Schorfheide (2004). This result shows that any parametrization in Lubik and Schorfheide (2004) has a unique mapping to our representation. In particular, we now consider the parametrization $\breve{M} = M^*(\theta) + M$, where M is centered at 0 and $M^*(\theta)$ is found by minimizing the distance between the impulse response functions under determinacy and indeterminacy at the boundary of the determinacy region. We can therefore write equation (35) as

$$\nu_{x,t} = \gamma_\varepsilon(M^*(\theta))\varepsilon_t + \gamma_\zeta\zeta_t, \tag{36}$$

where $\gamma_\varepsilon(M^*(\theta)) \equiv \left[\frac{\theta_y^2}{d^2}\frac{\theta_x}{(\theta_x-\theta_y)} + \frac{\theta_y}{d}M^*(\theta)\right]$ and $\gamma_\zeta \equiv \frac{\theta_y}{d}$. Given a parametrization $\{M^*(\theta), \sigma_\zeta\}$ and the normalization $E\left[\varepsilon_t\zeta_t\right] = 0$ in Lubik and Schorfheide (2004), we derive the corresponding variance and covariance terms of the non-fundamental shock, $\nu_{x,t}$, introduced in our approach as

$$\sigma^2_{\nu_x}(M^*(\theta)) = \gamma^2_\varepsilon(M^*(\theta))\sigma^2_\varepsilon + \gamma^2_\zeta\sigma^2_\zeta \tag{37}$$

$$\sigma_{\varepsilon,\nu_x}(M^*(\theta)) = \gamma_\varepsilon(M^*(\theta))\sigma_\varepsilon^2 \tag{38}$$

The variance-covariance matrix of the shocks $\hat{\varepsilon}_t^x = \{\varepsilon_t, \nu_{x,t}\}'$ can be written as

$$\Omega_{\hat{\varepsilon}^x}(M^*(\theta)) \equiv \begin{bmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon,\nu_x}(M^*(\theta)) \\ \sigma_{\varepsilon,\nu_x}(M^*(\theta)) & \sigma_{\nu_x}^2(M^*(\theta)) \end{bmatrix}. \tag{39}$$

Implementing a Cholesky decomposition, the shocks $\hat{\varepsilon}_t^x = \{\varepsilon_t, v_t^x\}'$ can be written as

$$\hat{\varepsilon}_t^x = \begin{bmatrix} \varepsilon_t \\ \nu_{x,t} \end{bmatrix} = L(M^*(\theta))u_t \equiv \begin{bmatrix} \sigma_\varepsilon & 0 \\ \frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon} & \sqrt{\sigma_{\nu_x}^2(M^*(\theta)) - \left(\frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon}\right)^2} \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}, \tag{40}$$

where $Var(u_t) = I$ and $E(u_t) = 0$. Finally, the parametrization in Lubik and Schorfheide (2004) can be mapped to the solution we obtained in equation (32) as

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \begin{bmatrix} 1 & \frac{\theta_x}{(\theta_y-\theta_x)} \\ 0 & 1 \\ 0 & \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ 0 & \theta_x \end{bmatrix} \begin{bmatrix} \sigma_\varepsilon & 0 \\ \frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon} & \sqrt{\sigma_{\nu_x}^2(M^*(\theta)) - \left(\frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon}\right)^2} \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}. \tag{41}$$

References

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