Supplementary Material

Supplement to "Recalcitrant betas: Intraday variation in the cross-sectional dispersion of systematic risk"

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This Online Supplement consists of two parts. The first part contains assumptions and proofs of the theoretical results in the paper. The second part provides additional empirical evidence.

Appendix S1: Assumptions and proofs

In the proofs, we denote with K a positive constant that does not depend on n and N, and can change from one line to another.

S1.1 *Assumptions*

Assumption A. *For the processes* $(X^{(j)})_{j\geq 0}$ *we have*:

(*a*) *For a sequence of stopping times,* $(T_m)_{m\geq 1}$*, increasing to infinity, the processes* $(\alpha^{(j)})_{j\geq 0}$, $(\beta^{(j)})_{j\geq 1}$, $(\gamma^{(j)})_{j\geq 1}$, *and* $(\widetilde{\sigma}^{(j)})_{j\geq 1}$, *are all uniformly bounded on* $[0, T\wedge T_m]$.

(*b*) *The processes* $|\sigma_t^{(0)}|$ *and* $|\sigma_{t-}^{(0)}|$ *take positive values on* $[0, T]$.

(*c*) *For a sequence of stopping times,* $(T_m)_{m\geq 1}$*, increasing to infinity and a sequence of constants,* $(K_m)_{m\geq 1}$*, we have uniformly in* $j\geq 1$:

$$\mathbb{E}\Big[\sup_{s,t\in[0,T\wedge T_m]}\big|\sigma_t^{(0)}-\sigma_s^{(0)}\big|^2+\sup_{s,t\in[0,T\wedge T_m]}\big|\beta_t^{(j)}-\beta_s^{(j)}\big|^2+\sup_{s,t\in[0,T\wedge T_m]}\big|\widetilde{\sigma}_t^{(j)}-\widetilde{\sigma}_s^{(j)}\big|^2$$
$$+\sup_{s,t\in[0,T\wedge T_m]}\big|\gamma_t^{(j)}-\gamma_s^{(j)}\big|\big|\gamma_t^{(j)}-\gamma_s^{(j)}\big|^\top\Big]\leq K_m|t-s|,$$
$$\big|\mathbb{E}\big(\beta_{t\wedge T_m}^{(j)}-\beta_{s\wedge T_m}^{(j)}\big)\big|+\big|\mathbb{E}\big(\sigma_{t\wedge T_m}^{(0)}-\sigma_{s\wedge T_m}^{(0)}\big)\big|\leq K_m|t-s|,$$
$$\big|\mathbb{E}\big(\chi_{s\wedge T_m,t\wedge T_m}^{(1)}\chi_{s\wedge T_m,t\wedge T_m}^{(2)}\big)\big|\leq K_m|t-s|^2,$$

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for $\chi^{(1)}_{s,t}$ equal to $\beta^{(j)}_t - \beta^{(j)}_s$ or $\sigma^{(0)}_t - \sigma^{(0)}_s$, and $\chi^{(2)}_{s,t}$ equal to one of $(W^{(0)}_t - W^{(0)}_s)^2 - (t-s)$, $(W^{(0)}_t - W^{(0)}_s)(B_t - B_s)$ and $(W^{(0)}_t - W^{(0)}_s)(\widetilde{W}^{(j)}_t - \widetilde{W}^{(j)}_s)$.

(d) We have $\sum_{s\le t}\Delta X^{(j)}_s = \int_0^t\int_E \delta^{(j)}(s,u)\mu(ds,du)$, for $j\ge 0$, and where μ is a Poisson random measure on $\mathbb{R}_+\times E$ with compensator $ds\otimes\nu(du)$, for some σ-finite measure ν on a Polish space E. Furthermore, the jump size functions $\delta^{(j)}$ are mappings $\Omega\times\mathbb{R}_+\times E\to\mathbb{R}$ which are locally predictable. For $j=0,1,\ldots,$ we have $\int_0^T\int_E 1(\delta^{(j)}(s,u)\neq 0)\,ds\nu(du)<\infty$. For a sequence of stopping times, $(T_m)_{m\ge 1}$, increasing to infinity and a sequence of nonnegative functions $(\Gamma_m(u))_{m\ge1}$ satisfying $\int_E(1\vee\Gamma_m^2(u))\nu(du)<\infty$, we have $|\delta^{(j)}(s,u)|\le\Gamma_m(u)$, uniformly in $j\ge0$, for $s\in[0,T\wedge T_m]$.

ASSUMPTION B. *We have the following uniform convergence in probability as $N\to\overline{N}$ for some $\overline{N}\in(0,\infty]$:*

$$\sup_{t=1,\ldots,T,\kappa\in[0,1]}\left|\frac{6}{N}\sum_{j=1}^N\left[\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\left(\widetilde{\sigma}^{(j)}_{t-1+\kappa}\right)^2}{\left(\sigma^{(0)}_{t-1+\kappa}\right)^2}\right]-\psi^{(a)}_{t,\kappa}\right|\xrightarrow{\mathbb{P}}0,$$

$$\sup_{t=1,\ldots,T,\kappa\in[0,1]}\left|4\left(\frac{1}{N}\sum_{j=1}^N\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\right)^2-\psi^{(b)}_{t,\kappa}\right|\xrightarrow{\mathbb{P}}0,$$

$$\sup_{t=1,\ldots,T,\kappa\in[0,1]}\left\|\frac{6}{N}\sum_{j=1}^N\left[\left(\beta^{(j)}_{t-1+\kappa}-1\right)\frac{\gamma^{(j)}_{t-1+\kappa}}{\sigma^{(0)}_{t-1+\kappa}}\right]-\Lambda_{t,\kappa}\right\|\xrightarrow{\mathbb{P}}0,$$

for some $\psi^{(a)}_{t,\kappa}$, $\psi^{(b)}_{t,\kappa}$ and $\Lambda_{t,\kappa}$, which are càdlàg functions of $t-1+\kappa$, and $\|\cdot\|$ denoting the Frobenius norm of a matrix. We further set $\psi^{(c)}_{t,\kappa}=\Lambda_{t,\kappa}\Lambda_{t,\kappa}^\top$.

To state the next assumption, we introduce the following notation:

$$\begin{aligned}
k^N_{t,\kappa}(z,u) &= -\frac{1}{N}\sum_{j=1}^N e^{i(u+z)\beta^{(j)}_{t-1+\kappa}}uz\Bigg[\left(\beta^{(j)}_{t-1+\kappa}-1\right)^4+\frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\gamma^{(j)}_{t-1+\kappa}\left(\gamma^{(j)}_{t-1+\kappa}\right)^\top}{\left(\sigma^{(0)}_{t-1+\kappa}\right)^2}\\
&\qquad+\frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\left(\widetilde{\sigma}^{(j)}_{t-1+\kappa}\right)^2}{\left(\sigma^{(0)}_{t-1+\kappa}\right)^2}\Bigg],\\
c^N_{t,\kappa}(z,u) &= \frac{1}{N}\sum_{j=1}^N e^{i(z-u)\beta^{(j)}_{t-1+\kappa}}uz\Bigg[\left(\beta^{(j)}_{t-1+\kappa}-1\right)^4+\frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\gamma^{(j)}_{t-1+\kappa}\left(\gamma^{(j)}_{t-1+\kappa}\right)^\top}{\left(\sigma^{(0)}_{t-1+\kappa}\right)^2}\\
&\qquad+\frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\left(\widetilde{\sigma}^{(j)}_{t-1+\kappa}\right)^2}{\left(\sigma^{(0)}_{t-1+\kappa}\right)^2}\Bigg].
\end{aligned}$$

ASSUMPTION C. *We have the following convergence in probability as $N\to\overline{N}$, with some $\overline{N}\in(0,\infty]$:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|k^N_{t,\kappa}(z,u)-k_{t,\kappa}(z,u)\right|^2w(u)w(z)\,du\,dz\xrightarrow{\mathbb{P}}0,$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|c^{N}_{t,\kappa}(z,u)-c_{t,\kappa}(z,u)\right|^2 w(u)w(z)\,du\,dz \xrightarrow{\mathbb{P}} 0.$$

for some random functions $k_{t,\kappa}(z,u)$ and $c_{t,\kappa}(z,u)$, some $t \in \mathbb{N}_+ \cap [0,T]$ and $\kappa \in [0,1]$, and with w being the weight function for the $\mathcal{L}^2(w)$ space defined in equation (10) *in the paper.*

S1.2 *Localization*

ASSUMPTION SA. *We have Assumption A with $T_1 = \infty$. Furthermore, the processes $(\alpha^{(j)})_{j\geq 0}$, $(\beta^{(j)})_{j\geq 1}$, $(\gamma^{(j)})_{j\geq 1}$, and $(\widetilde{\sigma}^{(j)})_{j\geq 1}$ are uniformly bounded on $[0,T]$, and $|\sigma^{(0)}_t|$ is bounded from below by a positive constant on $[0,T]$.*

We will prove the results under the stronger Assumption SA. A standard localization argument then can be used to show that they continue to hold under the weaker Assumption A.

S1.3 *Notation and decomposition*

We start with introducing some notation that will be used in the proofs. Throughout, we will use the shorthand notation $\mathbb{E}^n_{t,i}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{t-1+(i-1)/n})$ for $t \in \mathbb{N}_+$. For a generic process Z_t, we set

$$Z_{t,\kappa,n} = Z_{t-1+\frac{\lfloor \kappa n\rfloor - k_n}{n}}, \quad t \in \mathbb{N}_+, \kappa \in [0,1].$$

and

$$\Delta^{n,2}_{t,i} Z = Z_{(t-1)+i/n} - Z_{(t-1)+(i-2)/n}, \quad t \in \mathbb{N}_+, i = 2,\dots,n.$$

We denote the spot variances of the asset prices by

$$V^{(0)}_t = \big(\sigma^{(0)}_t\big)^2, \qquad V^{(j)}_t = \big(\beta^{(j)}_t\big)^2 V^{(0)}_t + \gamma^{(j)}_t\big(\gamma^{(j)}_t\big)^\top + \big(\widetilde{\sigma}^{(j)}_t\big)^2, \quad j = 1,\dots,N,$$

and their continuous martingale components by

$$X^{c,(j)}_t = \int_0^t \beta^{(j)}_s \sigma^{(0)}_s\, dW^{(0)}_s + 1_{\{j\geq 1\}}\int_0^t \gamma^{(j)}_s\, dB_s + 1_{\{j\geq 1\}}\int_0^t \widetilde{\sigma}^{(j)}_s\, d\widetilde{W}^{(j)}_s, \quad k = 0,1,\dots,N,$$

where we used the normalization $\beta^{(0)}_s = 1$, for $s \in [0,T]$. We further set for $j = 1,\dots,N$:

$$C^{(j)}_t = \big(\beta^{(j)}_t - 1\big)V^{(0)}_t, \quad t \in \mathbb{R}_+,$$

and for $t \in \mathbb{N}_+$ and $\kappa \in [0,1]$:

$$\widetilde{C}^{(j)}_{t,\kappa} = \frac{n}{|\widetilde{\mathcal{I}}^n_\kappa|}\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\int_{t-1+(i-1)\Delta_n}^{t-1+i\Delta_n}\big(\beta^{(j)}_s - 1\big)V^{(0)}_s\, ds,$$

$$\widetilde{V}^{(j)}_{t,\kappa} = \frac{n}{|\mathcal{I}^n_\kappa|}\sum_{i\in\mathcal{I}^n_\kappa}\int_{t-1+(i-1)\Delta_n}^{t-1+i\Delta_n} V^{(j)}_s\, ds.$$

as well as

$$\overline{C}_{t,\kappa}^{(j)}(2) = \frac{n}{2|\widetilde{\mathcal{I}}_\kappa^n|}\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\big[(\Delta_{t,i}^{n,2}X^{c,(j)} - \Delta_{t,i}^{n,2}X^{c,(0)})\Delta_{t,i}^{n,2}X^{c,(0)}\big],$$

$$\overline{V}_{t,\kappa}^{(j)} = \frac{n}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\big(\Delta_{t,i}^n X^{c,(j)}\big)^2.$$

We similarly define $\overline{\check{C}}_{t,\kappa}^{(j)}$ and $\overline{\check{V}}_{t,\kappa}^{(0)}$ as well as $\overline{\check{C}}_{t,\kappa}^{(j)}$ and $\overline{\check{V}}_{t,\kappa}^{(0)}$ from $\check{C}_{t,\kappa}^{(j)}$ and $\check{V}_{t,\kappa}^{(0)}$. Finally, we denote for $t\in\mathbb{N}_+$ and $\kappa\in[0,1]$:

$$\begin{aligned}\overline{\eta}_{t,i}^{n,(j)} &= \sigma_{t,\kappa,n}^{(0)}\gamma_{t,\kappa,n}^{(j)} n\Delta_{t,i}^n W^{(0)}\Delta_{t,i}^n B + \sigma_{t,\kappa,n}^{(0)}\widetilde{\sigma}_{t,\kappa,n}^{(j)} n\Delta_{t,i}^n W^{(0)}\Delta_{t,i}^n\widetilde{W}^{(j)}\\ &\quad + \big(\beta_{t,\kappa,n}^{(j)}-1\big)V_{t,\kappa,n}^{(0)}\big(n\big(\Delta_{t,i}^n W^{(0)}\big)^2 - 1\big), \quad k=1,\ldots,N, \end{aligned} \tag{S1}$$

$$\begin{aligned}\overline{\eta}_{t,i}^{n,(j)}(2) &= \frac{1}{2}\sigma_{t,\kappa,n}^{(0)}\gamma_{t,\kappa,n}^{(j)} n\Delta_{t,i}^{n,2} W^{(0)}\Delta_{t,i}^{n,2} B + \frac{1}{2}\sigma_{t,\kappa,n}^{(0)}\widetilde{\sigma}_{t,\kappa,n}^{(j)} n\Delta_{t,i}^{n,2} W^{(0)}\Delta_{t,i}^{n,2}\widetilde{W}^{(j)}\\ &\quad + \frac{1}{2}\big(\beta_{t,\kappa,n}^{(j)}-1\big)V_{t,\kappa,n}^{(0)}\big(n\big(\Delta_{t,i}^{n,2} W^{(0)}\big)^2 - 2\big), \quad k=1,\ldots,N, \end{aligned} \tag{S2}$$

$$\overline{\eta}_{t,i}^{n,(0)} = V_{t,\kappa,n}^{(0)}\big(n\big(\Delta_{t,i}^n W^{(0)}\big)^2 - 1\big), \tag{S3}$$

and we use them to define the following processes:

$$\mathcal{C}_{t,\kappa}^{(j)}(2) = \frac{1}{|\widetilde{\mathcal{I}}_\kappa^n|}\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\overline{\eta}_{t,i}^{n,(j)}(2), \qquad \check{\mathcal{C}}_{t,\kappa}^{(j)} = \frac{1}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\overline{\eta}_{t,i}^{n,(j)}\big(1_{\{i\in\mathcal{O}_\kappa^n\}} - 1_{\{i\in\mathcal{E}_\kappa^n\}}\big), \tag{S4}$$

$$\mathcal{V}_{t,\kappa}^{(0)} = \frac{1}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\overline{\eta}_{t,i}^{n,(0)}, \qquad \check{\mathcal{V}}_{t,\kappa}^{(0)} = \frac{1}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\overline{\eta}_{t,i}^{n,(0)}\big(1_{\{i\in\mathcal{O}_\kappa^n\}} - 1_{\{i\in\mathcal{E}_\kappa^n\}}\big). \tag{S5}$$

S1.4 *Preliminary results*

We start with establishing some preliminary results about the moments of components of the differences $\widehat{C}_{t,\kappa}^{(j)}(2) - C_{t,\kappa,n}^{(j)}$ and $\widehat{V}_{t,\kappa}^{(j)} - V_{t,\kappa,n}^{(j)}$ as well as $\check{C}_{t,\kappa}^{(j)}$ and $\check{V}_{t,\kappa}^{(0)}$.

LEMMA S1. *Assume Assumption SA holds. For $j,l=0,1,\ldots,N$ and $p\geq 1$, we have*

$$\begin{aligned}&\mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n\big|\widehat{C}_{t,\kappa}^{(j)}(2) - \overline{C}_{t,\kappa}^{(j)}(2)\big|^p + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n\big|\widehat{V}_{t,\kappa}^{(j,l)} - \overline{V}_{t,\kappa}^{(j)}\big|^p\\ &\quad + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n\big|\check{C}_{t,\kappa}^{(j)} - \overline{\check{C}}_{t,\kappa}^{(j)}\big|^p + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n\big|\check{V}_{t,\kappa}^{(0,l)} - \overline{\check{V}}_{t,\kappa}^{(0)}\big|^p\\ &\leq K\big(k_n^{1-p}\Delta_n^{1+p(2\varpi-1)} \vee \Delta_n^{p+p(2\varpi-1)}\big), \quad t=1,\ldots,T,\ \kappa\in[0,1],\end{aligned}$$

for some positive constant K, which can depend on p and ϖ but does not depend on k.

PROOF OF LEMMA S1. We show only the bound for the terms involving $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\check{C}_{t,\kappa}^{(j)}$, with the result for the terms involving $\widehat{V}_{t,\kappa}^{(j,l)}$ and $\check{V}_{t,\kappa}^{(0,l)}$ established analogously. For $j,l=$

$0, 1, \ldots, N$, we can make the decompositions:

$$
\begin{aligned}
&\Delta_{t,i}^n X^{(j)} \Delta_{t,i}^n X^{(l)} 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}} - \Delta_{t,i}^n X^{c,(j)} \Delta_{t,i}^n X^{c,(l)} \\
&\quad = -\Delta_{t,i}^n X^{c,(j)} \Delta_{t,i}^n X^{c,(l)} 1_{\{(\mathcal{A}_{t,i}^{(j,l)})^c\}} \\
&\qquad + \Delta_{t,i}^n X^{d,(j)} \Delta_{t,i}^n X^{d,(l)} 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}} + \big(\Delta_{t,i}^n X^{c,(j)} \Delta_{t,i}^n X^{d,(l)} + \Delta_{t,i}^n X^{d,(j)} \Delta_{t,i}^n X^{c,(l)}\big) 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}}, \\
&\Delta_{t,i}^{n,2} X^{(j)} \Delta_{t,i}^{n,2} X^{(l)} 1_{\{\mathcal{A}_{t,i-1}^{(j,l)} \cap \mathcal{A}_{t,i}^{(j,l)}\}} - \Delta_{t,i}^{n,2} X^{c,(j)} \Delta_{t,i}^{n,2} X^{c,(l)} \\
&\quad = -\Delta_{t,i}^{n,2} X^{c,(j)} \Delta_{t,i}^{n,2} X^{c,(l)} 1_{\{(\mathcal{A}_{t,i-1}^{(j,l)})^c \cup (\mathcal{A}_{t,i}^{(j,l)})^c\}} + \Delta_{t,i}^{n,2} X^{d,(j)} \Delta_{t,i}^{n,2} X^{d,(l)} 1_{\{\mathcal{A}_{t,i-1}^{(j,l)} \cap \mathcal{A}_{t,i}^{(j,l)}\}} \\
&\qquad + \big(\Delta_{t,i}^{n,2} X^{c,(j)} \Delta_{t,i}^{n,2} X^{d,(l)} + \Delta_{t,i}^{n,2} X^{d,(j)} \Delta_{t,i}^{n,2} X^{c,(l)}\big) 1_{\{\mathcal{A}_{t,i-1}^{(j,l)} \cap \mathcal{A}_{t,i}^{(j,l)}\}}.
\end{aligned}
$$

Next, using the bounds for the continuous and jump components of Itô semimartingales in Section 2.1.5 of Jacod and Protter (2012), together with Hölder's inequality, we get for $p \geq 1$ and arbitrary small $\iota > 0$:

$$
\mathbb{E}_{t,i}^n\Big(\big|\Delta_{t,i}^n X^{c,(j)} \Delta_{t,i}^n X^{c,(l)}\big|^p 1_{\{(\mathcal{A}_{t,i}^{(j,l)})^c\}}\Big) \leq K \Delta_n^{p+1-\iota}, \tag{S6}
$$

$$
\mathbb{E}_{t,i}^n\Big(\big|\Delta_{t,i}^n X^{c,(j)} \Delta_{t,i}^n X^{d,(l)}\big|^p 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}}\Big) \leq K \Delta_n^{p/2+p\varpi+1-\iota}, \tag{S7}
$$

$$
\mathbb{E}_{t,i}^n\Big(\big|\Delta_{t,i}^n X^{d,(j)} \Delta_{t,i}^n X^{d,(l)}\big|^p 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}}\Big) \leq K \Delta_n^{1+2p\varpi}. \tag{S8}
$$

Combining these bounds, and using successive application of Burkholder–Gundy–Davis inequalities as well as inequality in means, the result of the lemma follows. □

LEMMA S2. *Assume Assumption SA holds. For $j = 0, 1, \ldots, N$, $t = 1, \ldots, T$ and $\kappa \in [0, 1]$, we have*

$$
\begin{aligned}
&\mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\overline{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}\big|^p + \mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\overline{V}_{t,\kappa}^{(j)} - \widetilde{V}_{t,\kappa}^{(j)}\big|^p + \mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\breve{\mathcal{C}}_{t,\kappa}^{(j)}\big|^p \\
&\quad + \mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\mathcal{C}_{t,\kappa}^{(j)}(2)\big|^p + \mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\mathcal{V}_{t,\kappa}^{(0)}\big|^p + \mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\breve{\mathcal{V}}_{t,\kappa}^{(0)}\big|^p \\
&\mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\overline{\breve{C}}_{t,\kappa}^{(j)} - \widetilde{\breve{C}}_{t,\kappa}^{(j)}\big|^p + \mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n \big|\overline{\breve{V}}_{t,\kappa}^{(0)} - \widetilde{\breve{V}}_{t,\kappa}^{(0)}\big|^p \leq K k_n^{-p/2}, \quad p \geq 2,
\end{aligned}
$$

for some positive constant K which can depend on p but does not depend on j. Furthermore, we have the following decompositions:

$$
\begin{aligned}
\overline{C}_{t,\kappa}^{(j)}(2) &= \widetilde{C}_{t,\kappa}^{(j)} + \mathcal{C}_{t,\kappa}^{(j)}(2) + R_{t,\kappa,n}^{(j)}, & \overline{V}_{t,\kappa}^{(0)} &= \widetilde{V}_{t,\kappa}^{(0)} + \mathcal{V}_{t,\kappa}^{(0)} + R_{t,\kappa,n}, \\
\overline{\breve{C}}_{t,\kappa}^{(j)} &= \widetilde{\breve{C}}_{t,\kappa}^{(j)} + \breve{\mathcal{C}}_{t,\kappa}^{(j)} + \breve{R}_{t,\kappa,n}^{(j)}, & \overline{\breve{V}}_{t,\kappa}^{(0)} &= \widetilde{\breve{V}}_{t,\kappa}^{(0)} + \breve{\mathcal{V}}_{t,\kappa}^{(0)} + \breve{R}_{t,\kappa,n},
\end{aligned}
$$

where for $\mathcal{R}_{t,\kappa,n} = R_{t,\kappa,n}^{(k)}$, $R_{t,\kappa,n}$, $\breve{R}_{t,\kappa,n}^{(k)}$ or $\breve{R}_{t,\kappa,n}$, we have

$$
\mathbb{E}_{t,\lfloor \kappa n\rfloor - k_n+1}^n |\mathcal{R}_{t,\kappa,n}|^p \leq \frac{K}{k_n^{p-2} n}, \quad p \geq 2,
$$

and for any bounded function $\omega : [0,1] \to \mathbb{R}_+$, we also have

$$\mathbb{E}\left|\sum_{s=k_n}^{n} \omega(s\Delta_n)\mathcal{R}_{t,s\Delta_n,n}\right|^2 \leq Kk_n.$$

Proof of Lemma S2. The first result of the lemma follows by (successive) use of Burkholder–Davis–Gundy inequality. We now show the remaining claims for $\overline{C}_{t,\kappa}^{(j)}(2)$ only, with the corresponding result for $\overline{\check{C}}_{t,\kappa}^{(j)}$, $\overline{V}_{t,\kappa}^{(0)}$ and $\overline{\check{V}}_{t,\kappa}^{(0)}$ being established in an analogous way. Using Itô's lemma, we have

$$n\big(\Delta_{t,i}^{n,2}X^{c,(j)} - \Delta_{t,i}^{n,2}X^{c,(0)}\big)\Delta_{t,i}^{n,2}X^{c,(0)} = n\Delta_{t,i}^{n,2}C^{(j)} + \eta_{t,i}^{n,(j)}(2),$$

where the term $\eta_{t,i}^{n,(j)}(2)$ satisfies

$$\mathbb{E}_{t,i-1}^{n}\big(\eta_{t,i}^{n,(j)}(2) - \overline{\eta}_{t,i}^{n,(k)}(2)\big) = 0, \qquad \mathbb{E}_{t,i-1}^{n}\big|\eta_{t,i}^{n,(j)}(2) - \overline{\eta}_{t,i}^{n,(j)}(2)\big|^p \leq Kk_n/n, \quad p \geq 2,$$

and for the second result we made use of the smoothness in expectation assumption for the processes $\sigma_t^{(0)}$, $\beta_t^{(j)}$, $\gamma_t^{(j)}$ and $\widetilde{\sigma}_t^{(j)}$ as well as Burkholder–Davis–Gundy inequality. From here, the result of the lemma follows. □

Lemma S3. *Assume Assumption SA holds. For $j = 0, 1, \ldots, N$, $t = 1, \ldots, T$, and $\kappa \in [0,1]$, we have*

$$\begin{aligned}
&\mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big(\big|\widetilde{C}_{t,\kappa}^{(j)} - C_{t,\kappa,n}^{(j)}\big|^p 1_{\{k\neq 0\}}\big) + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big|\widetilde{V}_{t,\kappa}^{(j)} - V_{t,\kappa,n}^{(j)}\big|^p \\
&\quad + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big(\big|\check{\widetilde{C}}_{t,\kappa}^{(j)}\big|^p 1_{\{k\neq 0\}}\big) + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big|\check{\widetilde{V}}_{t,\kappa}^{(j)}\big|^p \leq Kk_n/n, \quad p \geq 2,
\end{aligned}$$

for some positive constant K which can depend on p but does not depend on j. In addition, for $\omega : [0,1] \to \mathbb{R}_+$ that is Lipschitz continuous and $t = 1, \ldots, T$, we have

$$\frac{1}{n-k_n+1}\sum_{s=k_n}^{n}\left[\omega(s\Delta_n)\left(\frac{\widetilde{C}_{t,s\Delta_n}^{(j)}}{\widetilde{V}_{t,s\Delta_n}^{(0)}}\right)^2\right] = \int_{t-1}^{t}\omega(s-t+1)\big(\beta_s^{(j)}-1\big)^2\,ds + O_p\left(\frac{k_n}{n}\right),$$

with $j = 1, \ldots, N$.

Proof of Lemma S3. Using Assumption SA regarding the smoothness in expectation of the processes $\beta_t^{(j)}$ and $\sigma_t^{(0)}$ as well as the boundedness of these processes, we can write

$$\begin{aligned}
&\big|\mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big(\widetilde{C}_{t,\kappa}^{(j)} - C_{t,\kappa,n}^{(j)}\big)\big| + \big|\mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big(\widetilde{V}_{t,\kappa}^{(j)} - V_{t,\kappa,n}^{(j)}\big)\big| \leq Kk_n/n, \\
&\mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big|\widetilde{C}_{t,\kappa}^{(j)} - C_{t,\kappa,n}^{(j)}\big|^p + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^{n}\big|\widetilde{V}_{t,\kappa}^{(j)} - V_{t,\kappa,n}^{(j)}\big|^p \leq Kk_n/n, \quad p \geq 2,
\end{aligned}$$

where the constant K does not depend on κ. From here, the results of the lemma follow directly by taking into account that ω is Lipschitz continuous. □

Lemma S4. *Assume Assumption SA holds and* $k_n\Delta_n^{2-4\varpi}\to\infty$. *For* $j,l=0,1,\ldots,N$ *and* $p\geq 1$, *we have*

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\left|\frac{1}{|\mathcal{I}^n_\kappa|}\sum_{i\in\mathcal{I}^n_\kappa}\big(\Delta^n_{t,i}X^{(0)}\Delta^n_{t,i}X^{(j)}1_{\{\mathcal{A}^{(0,l)}_{t,i}\}}\big)^2\right|^p\leq K,\quad t=1,\ldots,T,\ \kappa\in[0,1].$$

Proof of Lemma S4. The result of the lemma follows by application of the bounds in (S6)–(S8) together with an application of Burkholder–Davis–Gundy inequalities, and further taking into account that $k_n\Delta_n^{2-4\varpi}\to\infty$. □

S1.5 *Proof of Theorem 1*

Second-order Taylor expansion yields for $j=1,\ldots,N$:

$$\begin{aligned}&\left(\frac{\widehat{C}^{(j)}_{t,\kappa}(2)}{\widehat{V}^{(0,j)}_{t,\kappa}}\right)^2 1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}-\left(\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\widetilde{V}^{(0)}_{t,\kappa}}\right)^2\\&\quad=Z^{(j)}_{t,\kappa,n}+B^{(j)}_{t,\kappa,n}-\left(\left(\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\widetilde{V}^{(0)}_{t,\kappa}}\right)^2+Z^{(j)}_{t,\kappa,n}+B^{(j)}_{t,\kappa,n}\right)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}}+\overline{R}^{(j)}_{t,\kappa,n},\end{aligned}$$

where

$$\begin{aligned}Z^{(j)}_{t,\kappa,n}&=2\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\widetilde{V}^{(0)}_{t,\kappa}}\frac{1}{\widetilde{V}^{(0)}_{t,\kappa}}\left(\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}-\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\widetilde{V}^{(0)}_{t,\kappa}}\big(\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big)\right),\\B^{(j)}_{t,\kappa,n}&=3\frac{\big(\widetilde{C}^{(j)}_{t,\kappa}\big)^2}{\big(\widetilde{V}^{(0)}_{t,\kappa}\big)^4}\big(\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big)^2+\frac{1}{\big(\widetilde{V}^{(0)}_{t,\kappa}\big)^2}\big(\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}\big)^2\\&\quad-4\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\big(\widetilde{V}^{(0)}_{t,\kappa}\big)^3}\big(\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big)\big(\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}\big),\end{aligned}\tag{S9}$$

and the residual term $\overline{R}^{(j)}_{t,\kappa,n}$ satisfies

$$\begin{aligned}\big|\overline{R}^{(j)}_{t,\kappa,n}\big|&\leq K\alpha_n^{-4}\big(\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|^3\vee\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|^6\big)\\&\quad+K\alpha_n^{-4}\big|\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}\big|\big(\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|^2\vee\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|^4\big)\\&\quad+K\alpha_n^{-4}\big|\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}\big|^2\big(\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|\vee\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|^4\big),\end{aligned}\tag{S10}$$

where the constant K does not depend on j. Finally, in what follows we use the following notation:

$$\begin{aligned}\overline{Z}^{(j)}_{t,\kappa,n}&=2\big(\beta^{(j)}_{t,\kappa,n}-1\big)^2\frac{n}{|\widetilde{\mathcal{I}}^n_\kappa|}\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\big[\Delta^n_{t,i-1}W^{(0)}\Delta^n_{t,i}W^{(0)}\big]\\&\quad+\big(\beta^{(j)}_{t,\kappa,n}-1\big)\frac{n}{|\widetilde{\mathcal{I}}^n_\kappa|}\end{aligned}$$

$$\times \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\left[\frac{\gamma^{(j)}_{t,\kappa,n}}{\sigma^{(0)}_{t,\kappa,n}}\Delta^{n,2}_{t,i}W^{(0)}\Delta^{n,2}_{t,i}B+\frac{\widetilde{\sigma}^{(j)}_{t,\kappa,n}}{\sigma^{(0)}_{t,\kappa,n}}\Delta^{n,2}_{t,i}W^{(0)}\Delta^{n,2}_{t,i}\widetilde{W}^{(j)}\right]. \tag{S11}$$

The proof of the theorem consists of a sequence of lemmas.

LEMMA S5. *Under Assumption SA, and provided $k_n\Delta_n\to 0$ and $k_n\Delta_n^{1-2\varpi}\to\infty$, we have for $t=1,\dots,T$ and $\kappa\in[0,1]$:*

$$\frac{1}{N}\sum_{j=1}^N\overline{R}^{(j)}_{t,\kappa,n}+\frac{1}{n-k_n+1}\sum_{s=k_n}^n\frac{1}{N}\sum_{j=1}^N\left|\overline{R}^{(j)}_{t,s\Delta_n,n}\right|=O_p\left(\frac{1}{\alpha_n^4}\left(k_n^{-2}\Delta_n^{1+3(2\varpi-1)}\vee\frac{1}{k_n^{3/2}}\right)\right).$$

PROOF OF LEMMA S5. The result of the lemma follows by an application of Hölder's inequality and making use of the bounds of Lemmas S1–S2 as well as the inequality in (S10). □

LEMMA S6. *Under Assumption SA, and provided $\varpi\in(1/4,1/2)$, $\varrho\in(0,1/2)$ and $k_n\Delta_n^{2-4\varpi}\to\infty$, we have for $t=1,\dots,T$, $\kappa\in[0,1]$ and some arbitrary small $\iota>0$:*

$$\widehat{B}^N_{t,\kappa}-\frac{1}{N}\sum_{j=1}^N B^{(j)}_{t,\kappa,n}=O_p\left(\frac{1}{\alpha_n^4}\left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\vee\left(\frac{k_n}{n}\right)^{1-\iota}\vee\frac{1}{k_n^{3/2}}\right)\right), \tag{S12}$$

and further for any bounded function $\omega:[0,1]\to\mathbb{R}$:

$$\begin{aligned}\frac{1}{n-k_n+1}&\sum_{s=k_n}^n\left[\omega(s\Delta_n)\left(\widehat{B}^N_{t,s\Delta_n}-\frac{1}{N}\sum_{j=1}^N B^{(j)}_{t,s\Delta_n,n}\right)\right]\\&=O_p\left(\frac{1}{\alpha_n^4}\left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\vee\left(\frac{k_n}{n}\right)^{1-\iota}\vee\frac{1}{k_n^{3/2}}\right)\right).\end{aligned} \tag{S13}$$

PROOF OF LEMMA S6. We start with defining the processes for $j=1,\dots,N$ (recall the notation in equations (S1)–(S5)):

$$\mathcal{B}^{1,(j)}_{t,\kappa,n}=-\frac{\left(\beta^{(j)}_{t,\kappa,n}-1\right)^2}{\left(V^{(0)}_{t,\kappa,n}\right)^2}\left(\breve{\mathcal{V}}^{(0)}_{t,\kappa}\right)^2+\frac{3}{2\left(V^{(0)}_{t,\kappa,n}\right)^2}\left(\breve{\mathcal{C}}^{(j)}_{t,\kappa}\right)^2,$$

$$\mathcal{B}^{2,(j)}_{t,\kappa,n}=3\frac{\left(\beta^{(j)}_{t,\kappa,n}-1\right)^2}{\left(V^{(0)}_{t,\kappa,n}\right)^2}\left(\mathcal{V}^{(0)}_{t,\kappa}\right)^2+\frac{1}{\left(V^{(0)}_{t,\kappa,n}\right)^2}\left(\mathcal{C}^{(j)}_{t,\kappa}(2)\right)^2-4\frac{\left(\beta^{(j)}_{t,\kappa,n}-1\right)}{\left(V^{(0)}_{t,\kappa,n}\right)^2}\mathcal{V}^{(0)}_{t,\kappa}\mathcal{C}^{(j)}_{t,\kappa}(2).$$

By direct calculation,

$$\begin{aligned}\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\left(\breve{\mathcal{C}}^{(j)}_{t,\kappa}\right)^2&=\frac{1}{\left|\mathcal{I}^n_\kappa\right|}V^{(0)}_{t,\kappa,n}\gamma^{(j)}_{t,\kappa,n}\left(\gamma^{(j)}_{t,\kappa,n}\right)^\top+\frac{1}{\left|\mathcal{I}^n_\kappa\right|}V^{(0)}_{t,\kappa,n}\left(\widetilde{\sigma}^{(j)}_{t,\kappa,n}\right)^2\\&\quad+\frac{1}{\left|\mathcal{I}^n_\kappa\right|}\left(V^{(0)}_{t,\kappa,n}\right)^2\left(\beta^{(j)}_{t,\kappa,n}-1\right)^2,\end{aligned}$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{C}^{(j)}_{t,\kappa}(2)\big)^2 = \frac{3}{2}\frac{|\mathcal{I}^n_\kappa|}{|\widetilde{\mathcal{I}}^n_\kappa|}\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\breve{\mathcal{C}}^{(j)}_{t,\kappa}\big)^2,$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{V}^{(0)}_{t,\kappa}\big)^2 = \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\breve{\mathcal{V}}^{(0)}_{t,\kappa}\big)^2 = \frac{2}{|\mathcal{I}^n_\kappa|}\big(V^{(0)}_{t,\kappa,n}\big)^2,$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{V}^{(0)}_{t,\kappa}\mathcal{C}^{(j)}_{t,\kappa}(2)\big) = \big(\beta^{(j)}_{t,\kappa,n}-1\big)\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{V}^{(0)}_{t,\kappa}\big)^2.$$

From here, using the first bound in Lemma S2, we have

$$\big|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{B}^{1,(j)}_{t,\kappa,n}-\mathcal{B}^{2,(j)}_{t,\kappa,n}\big)\big| \le \frac{K}{k_n^2},\ \mathbb{E}\big[\big(\mathcal{B}^{1,(j)}_{t,\kappa,n}\big)^2+\big(\mathcal{B}^{2,(j)}_{t,\kappa,n}\big)^2\big] \le \frac{K}{k_n^2}, \tag{S14}$$

where the constant K does not depend on t, κ and j. Next, using Lemmas S1–S2, Cauchy–Schwarz and Burkholder–Davis–Gundy inequalities and taking into account that $\varpi > 1/4$ and $\varrho \in (0,1/2)$, we have

$$\mathbb{E}\big|B^{(j)}_{t,\kappa,n}-\mathcal{B}^{2,(j)}_{t,\kappa,n}\big| \le K\bigg(\frac{\Delta_n^{1+2(2\varpi-1)}}{k_n}\vee\frac{1}{\sqrt{k_n n}}\bigg).$$

We proceed with analyzing the difference $\widehat{B}^N_{t,\kappa,n}-\frac{1}{N}\sum_{j=1}^N\mathcal{B}^{1,(j)}_{t,\kappa,n}$. First, using Lemma S1–S4, Hölder's inequality as well as the restriction $k_n\Delta_n^{2-4\varpi}\to\infty$ of the lemma, we have

$$\mathbb{E}\big|\widehat{V}^{(j)}_{\beta,t,\kappa}\big(\breve{V}^{(0,j)}_{t,\kappa}\big)^2 1_{\{\breve{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\big| \le \frac{K}{\alpha_n^2}\frac{1}{k_n}\bigg[\frac{1}{k_n}\vee\Delta_n^{4\varpi-\iota}\bigg],$$

for some arbitrary small $\iota>0$. We continue with introducing the following notation:

$$\xi^{(j,1)}_{t,\kappa} = \frac{\widehat{\beta}^{(j)}_{t,\kappa}-1}{\widehat{V}^{(0,j)}_{t,\kappa}}\breve{V}^{(0)}_{t,\kappa} - \frac{\beta^{(j)}_{t,\kappa,n}-1}{V^{(0)}_{t,\kappa,n}}\breve{\mathcal{V}}^{(0)}_{t,\kappa},$$

$$\xi^{(j,2)}_{t,\kappa} = \frac{1}{\widehat{V}^{(0,j)}_{t,\kappa}}\breve{C}^{(j)}_{t,\kappa} - \frac{1}{V^{(0)}_{t,\kappa,n}}\breve{\mathcal{C}}^{(j)}_{t,\kappa}.$$

Then we have

$$\begin{aligned}\big|\xi^{(j,1)}_{t,\kappa}\big|1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}} \le{}& \frac{K}{\alpha_n^2}\big(\big|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}\big|\vee\big|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}\big|\vee 1\big)\big|\breve{V}^{(0)}_{t,\kappa}-\breve{\mathcal{V}}^{(0)}_{t,\kappa}\big| \\ &+\frac{K}{\alpha_n^2}\big|\breve{\mathcal{V}}^{(0)}_{t,\kappa}\big|\big(\big|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}\big|\vee\big|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}\big|^2\vee\big|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}\big|\big),\\ \big|\xi^{(j,2)}_{t,\kappa}\big|1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}} \le{}& \frac{K}{\alpha_n}\big|\breve{C}^{(j)}_{t,\kappa}-\breve{\mathcal{C}}^{(j)}_{t,\kappa}\big|+\frac{K}{\alpha_n}\big|\breve{\mathcal{C}}^{(j)}_{t,\kappa}\big|\big|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}\big|,\end{aligned}$$

where we denote

$$\begin{aligned}\widehat{C}^{(j,0)}_{t,\kappa} &= \frac{n}{|\mathcal{I}^n_\kappa|}\sum_{i\in\mathcal{I}^n_\kappa}\Delta^n_{t,i}X^{(j)}\Delta^n_{t,i}X^{(0)}1_{\{\mathcal{A}^{(j,0)}_{t,i}\}},\\ C^{(j,0)}_t &= \int_0^t\beta^{(j)}_s V^{(0)}_s\,ds,\quad j=1,\dots,N.\end{aligned} \tag{S15}$$

From here, using Lemmas S1–S3, we have

$$\mathbb{E}^n_{t,\lfloor \kappa n\rfloor-k_n+1}\big[\big(|\xi^{(j,1)}_{t,\kappa}|^2+|\xi^{(j,2)}_{t,\kappa}|^2\big)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\big]\leq \frac{K}{\alpha_n^4}\bigg(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\vee\frac{1}{k_n^2}\vee\Big(\frac{k_n}{n}\Big)^{1-\iota}\bigg),$$

for some arbitrary small $\iota>0$. Similarly, we have

$$\mathbb{E}^n_{t,\lfloor \kappa n\rfloor-k_n+1}\big[\big(|\xi^{(j,1)}_{t,\kappa}\|\check{\mathcal{V}}^{(0)}_{t,\kappa}|+|\xi^{(j,2)}_{t,\kappa}\|\check{\mathcal{C}}^{(j)}_{t,\kappa}|\big)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\big]\leq \frac{K}{\alpha_n^2}\bigg(\frac{1}{k_n^{3/2}}\vee\Big(\frac{k_n}{n}\Big)^{1-\iota}\frac{\Delta_n^{2\varpi-1}}{k_n^2}\bigg).$$

Furthermore, given the lower bound restriction on $V_t^{(0)}$ in Assumption SA, we have

$$1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}}\leq K_p|\widehat{V}^{(0)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}|^p,\quad \forall p\geq 1,$$

where the constant K_p depends on p. Therefore, by making use again of Lemmas S1–S3 and taking into account the restriction on ϖ and ϱ of the lemma, we have altogether

$$\mathbb{E}\bigg|\widehat{B}^N_{t,\kappa,n}-\frac{1}{N}\sum_{j=1}^N\mathcal{B}^{1,(j)}_{t,\kappa,n}\bigg|\leq\frac{K}{\alpha_n^4}\bigg(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\vee\frac{1}{k_n^{3/2}}\vee\Big(\frac{k_n}{n}\Big)^{1-\iota}\bigg),$$

where again the constant K does not depend on t, κ and j. Combining the above bounds, we get the first bound of the lemma in (S12). For the second bound in (S13), we make in addition use of the following:

$$\frac{1}{n^2}\sum_{s=k_n}^n\big[\omega^2(s\Delta_n)\big(\mathcal{B}^{1,(j)}_{t,s\Delta_n,n}-\mathcal{B}^{2,(j)}_{t,s\Delta_n,n}-\mathbb{E}^n_{t,s-k_n+1}\big(\mathcal{B}^{1,(j)}_{t,s\Delta_n,n}-\mathcal{B}^{2,(j)}_{t,s\Delta_n,n}\big)\big)^2\big]\leq\frac{K}{k_n n},$$

which in turn follows by application of Cauchy–Schwarz inequality and the second bound in (S14). □

Lemma S7. *Under Assumption SA, and provided $\varrho\in(0,1/2)$ we have for $t=1,\ldots,T$ and $\kappa\in[0,1]$ as well as some arbitrary small $\iota>0$:*

$$\big|\mathbb{E}^n_{t,\lfloor \kappa n\rfloor-k_n+1}\big(Z^{(j)}_{t,\kappa,n}-\overline{Z}^{(j)}_{t,\kappa,n}\big)\big|\leq K\bigg(\Delta_n^{2\varpi}\vee\frac{k_n}{n}\vee\Big(\frac{k_n}{n}\Big)^{1-\iota}\frac{1}{\sqrt{k_n}}\bigg),$$

$$\mathbb{E}^n_{t,\lfloor \kappa n\rfloor-k_n+1}\big(Z^{(j)}_{t,\kappa,n}-\overline{Z}^{(j)}_{t,\kappa,n}\big)^2\leq K\bigg(\Delta_n^{2\varpi}\vee\Big(\frac{k_n}{n}\Big)^{1-\iota}\frac{1}{k_n}\vee\frac{1}{k_n^2}\bigg),$$

where the constant K does not depend on t, κ, and j, and $Z^{(j)}_{t,\kappa,n}$ and $\overline{Z}^{(j)}_{t,\kappa,n}$ are defined in (S9) and (S11), respectively.

Proof of Lemma S7. We denote with $\mathcal{Z}^{(j)}_{t,\kappa,n}$, the counterpart of $Z^{(j)}_{t,\kappa,n}$ in which $\widehat{C}^{(j)}_{t,\kappa}(2)$ and $\widehat{V}^{(0,j)}_{t,\kappa}$ are replaced with $\overline{C}^{(j)}_{t,\kappa}(2)$ and $\overline{V}^{(0)}_{t,\kappa}$, respectively. Then, using Lemma S1 and taking into account that $\varrho\in(0,1/2)$, we have

$$\mathbb{E}^n_{t,\lfloor \kappa n\rfloor-k_n+1}\big|Z^{(j)}_{t,\kappa,n}-\mathcal{Z}^{(j)}_{t,\kappa,n}\big|+\mathbb{E}^n_{t,\lfloor \kappa n\rfloor-k_n+1}\big|Z^{(j)}_{t,\kappa,n}-\mathcal{Z}^{(j)}_{t,\kappa,n}\big|^2\leq K\Delta_n^{2\varpi},$$

where the constant K does not depend on t, κ and j. Using successive conditioning and Assumption SA, we have

$$\left|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\chi_{1,n}\chi_{2,n})\right|\leq K\frac{k_n}{n},$$

for

$$\chi_{1,n}=\widetilde{C}^{(j)}_{t,\kappa}-C^{(j)}_{t,\kappa,n}\quad\text{or}\quad \widetilde{V}^{(0)}_{t,\kappa}-V^{(0)}_{t,\kappa,n},\qquad \chi_{2,n}=\overline{C}^{(j)}_{t,\kappa,n}(2)-\widetilde{C}^{(j)}_{t,\kappa,n}\quad\text{or}\quad \overline{V}^{(0)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa,n}.$$

From here, using Taylor expansion and Lemmas S2–S3, we have

$$\left|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{Z}^{(j)}_{t,\kappa,n}-\overline{Z}^{(j)}_{t,\kappa,n}\big)\right|\leq K\left(\frac{k_n}{n}\vee\left(\frac{k_n}{n}\right)^{1-\iota}\frac{1}{\sqrt{k_n}}\right).$$

Similar analysis leads to

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\mathcal{Z}^{(j)}_{t,\kappa,n}-\overline{Z}^{(j)}_{t,\kappa,n}\big)^2\leq K\left(\left(\frac{k_n}{n}\right)^{1-\iota}\frac{1}{k_n}\vee\frac{1}{k_n^2}\right).$$

From here, the results of the lemma follow. □

Lemma S8. *Under Assumption SA, and provided $k_n\Delta_n\to 0$ and $k_n\Delta_n^{1-2\varpi}\to\infty$, we have for $t=1,\ldots,T$, $\kappa\in[0,1]$, and $j=1,\ldots,N$:*

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\left[\big(1+|Z^{(j)}_{t,\kappa,n}|+|B^{(j)}_{t,\kappa,n}|\big)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}}\right]\leq K\left(\frac{k_n}{n}\vee\frac{1}{k_n^{p/2}}\vee k_n^{1-p-\iota}\Delta_n^{1+p(2\varpi-1)-\iota}\right),$$

for some arbitrary big $p>2$ and some arbitrary small $\iota>0$, and where the constant K depends on p but not on j, t, and κ.

Proof of Lemma S8. Using the inequality $1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}}\leq K_p|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}|^p$, for arbitrary $p\geq 1$, we have

$$\begin{aligned}&\big(1+|Z^{(j)}_{t,\kappa,n}|+|B^{(j)}_{t,\kappa,n}|\big)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}}\\ &\quad\leq K\big|\widehat{V}^{(0,j)}_{t,\kappa}-\widetilde{V}^{(0)}_{t,\kappa}\big|^p\big(1+|\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}|+|\widehat{C}^{(j)}_{t,\kappa}(2)-\widetilde{C}^{(j)}_{t,\kappa}|^2\big),\end{aligned}$$

for some arbitrary $p>2$ and with the constant K not depending on j, t, and κ. From here, the result of the lemma follows by application of Lemmas S1–S3 as well as the assumed relative growth conditions for k_n. □

For the statement of the next lemma, we need some additional notation. We denote with $\widetilde{\text{Avar}}(\widehat{D}^N_{t,\kappa})$ the counterpart of $\widehat{\text{Avar}}(\widehat{D}^N_{t,\kappa})$ in which we replace $\widehat{\beta}^{(j)}_{t,\kappa}$ with $\beta^{(j)}_{t-1+\kappa}$, $\widehat{V}^{(0)}_{t,\kappa}$ with $V^{(0)}_{t-1+\kappa}$, $\check{C}^{(j)}_{t,\kappa}$ with $\check{\mathcal{C}}^{(j)}_{t,\kappa}$, and $\check{V}^{(0)}_{t,\kappa}$ with $\check{\mathcal{V}}^{(0)}_{t,\kappa}$. In addition, we set

$$\text{Avar}(\widehat{D}^N_{t,\kappa})=\frac{6}{N^2|\widetilde{\mathcal{I}}^n_\kappa|}\sum_{j=1}^N\left[\big(\beta^{(j)}_{t-1+\kappa}-1\big)^2\frac{\big(\widetilde{\sigma}^{(j)}_{t-1+\kappa}\big)^2}{\big(\sigma^{(0)}_{t-1+\kappa}\big)^2}\right]+\frac{4}{|\widetilde{\mathcal{I}}^n_\kappa|}\left(\frac{1}{N}\sum_{j=1}^N\big(\beta^{(j)}_{t-1+\kappa}-1\big)^2\right)^2$$

$$+\frac{6}{|\widetilde{\mathcal{I}}^n_\kappa|}\left(\frac{1}{N}\sum_{j=1}^N\frac{(\beta^{(j)}_{t-1+\kappa}-1)\gamma^{(j)}_{t-1+\kappa}}{\sigma^{(0)}_{t-1+\kappa}}\right)\left(\frac{1}{N}\sum_{j=1}^N\frac{(\beta^{(j)}_{t-1+\kappa}-1)(\gamma^{(j)}_{t-1+\kappa})^\top}{\sigma^{(0)}_{t-1+\kappa}}\right).$$

LEMMA S9. *Under Assumption SA, and provided $\varpi\in(0,1/2)$, $\varrho\in(0,1/2)$, and $k_n\Delta_n^{2-4\varpi}\to\infty$, we have for $t=1,\ldots,T$, and $\kappa\in[0,1]$ and some arbitrary small $\iota>0$:*

$$\begin{aligned}
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big|\widehat{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa})-\mathrm{Avar}(\widehat{D}^N_{t,\kappa})\big| \\
&\quad\le\frac{K}{\alpha_n^6}\left(\frac{1}{\sqrt{n}}\vee\frac{1}{k_n^{3/2}}\vee\frac{\Delta_n^{2\varpi-\iota}}{k_n}\vee\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\right),\\
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big(\widetilde{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa})-\mathrm{Avar}(\widehat{D}^N_{t,\kappa})\big)=0,\\
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big((\widetilde{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa}))-\mathrm{Avar}(\widehat{D}^N_{t,\kappa})\big)^2\le\frac{K}{k_n^3}.
\end{aligned}$$

PROOF OF LEMMA S9. We start with the first bound. If we denote

$$\begin{aligned}
\xi^{(j,3)}_{t,\kappa}&=\frac{\widehat{\beta}^{(j)}_{t,\kappa}-1}{\widehat{V}^{(0,j)}_{t,\kappa}}\check{C}^{(j)}_{t,\kappa}-\frac{\beta^{(j)}_{t,\kappa,n}-1}{V^{(0)}_{t,\kappa,n}}\check{\mathcal{C}}^{(j)}_{t,\kappa},\\
\xi^{(j,4)}_{t,\kappa}&=\frac{(\widehat{\beta}^{(j)}_{t,\kappa}-1)^2}{\widehat{V}^{(0,j)}_{t,\kappa}}\check{V}^{(0)}_{t,\kappa}-\frac{(\beta^{(j)}_{t,\kappa,n}-1)^2}{V^{(0)}_{t,\kappa,n}}\check{\mathcal{V}}^{(0)}_{t,\kappa},
\end{aligned}$$

then direct calculation shows

$$\begin{aligned}
&|\xi^{(j,3)}_{t,\kappa}|1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\\
&\quad\le\frac{K}{\alpha_n^2}\big(|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}|\vee|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}|\vee 1\big)|\check{C}^{(j)}_{t,\kappa}-\check{\mathcal{C}}^{(j)}_{t,\kappa}|\\
&\qquad+\frac{K}{\alpha_n^2}|\check{\mathcal{C}}^{(j)}_{t,\kappa}|\big(|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}|\vee|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}|^2\vee|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}|\big),\\
&|\xi^{(j,4)}_{t,\kappa}|1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\\
&\quad\le\frac{K}{\alpha_n^3}\big(|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}|^2\vee|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}|^2\vee 1\big)|\check{V}^{(0)}_{t,\kappa}-\check{\mathcal{V}}^{(0)}_{t,\kappa}|\\
&\qquad+\frac{K}{\alpha_n^3}|\check{\mathcal{V}}^{(0)}_{t,\kappa}|\big(|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}|^3\vee|\widehat{V}^{(0,j)}_{t,\kappa}-V^{(0)}_{t,\kappa,n}|\vee|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}|^2\vee|\widehat{C}^{(j,0)}_{t,\kappa}-C^{(j,0)}_{t,\kappa,n}|\big),
\end{aligned}$$

where we use the notation $\widehat{C}^{(j,0)}_{t,\kappa}$ and $C^{(j,0)}_t$ as defined in the proof of Lemma S6 (see equation (S15)). From here, using Lemmas S1–S3, we have

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\big[\big(|\xi^{(j,3)}_{t,\kappa}|^2+|\xi^{(j,4)}_{t,\kappa}|^2\big)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\big]\le\frac{K}{\alpha_n^6}\left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\vee\frac{1}{k_n^2}\vee\left(\frac{k_n}{n}\right)^{1-\iota}\right),$$

for some arbitrary small $\iota > 0$. Similarly, we have

$$\mathbb{E}^n_{t,\lfloor \kappa n\rfloor - k_n+1}\left[\left(\left|\xi^{(j,3)}_{t,\kappa}\right| + \left|\xi^{(j,4)}_{t,\kappa}\right|\right)\left(\left|\check{\mathcal{C}}^{(j)}_{t,\kappa}\right| + \left|\check{\mathcal{V}}^{(0)}_{t,\kappa}\right|\right)1_{\{\widehat{V}^{(0)}_{t,\kappa} > \alpha_n\}}\right] \le \frac{K}{\alpha_n^3}\left(\frac{1}{k_n^{3/2}} \vee \left(\frac{k_n}{n}\right)^{1-\iota}\frac{\Delta_n^{2\varpi-1}}{k_n^2}\right).$$

From here, using again Lemmas S1–S3, we get the first bound of the lemma. The second and third results of the lemma follow by direct calculation. □

For the statement of the next lemma, we introduce some additional notation. We denote

$$\widehat{Z}^{(a)}_{t,\kappa,n} = \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa} \chi^{(a)}_{i,t,\kappa,n}, \qquad \widehat{Z}^{(b)}_{t,\kappa,n} = \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa} \chi^{(b)}_{i,t,\kappa,n},$$

$$\widehat{Z}^{(c)}_{t,\kappa,n} = \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa} \chi^{(c)}_{i,t,\kappa,n}, \quad t = 1,\dots,T,\ \kappa\in[0,1],$$

where

$$\chi^{(a)}_{i,t,\kappa,n} = \frac{1}{\sqrt{N}}\frac{n}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\sum_{j=1}^N\left[\left(\beta^{(j)}_{t,\kappa,n}-1\right)\frac{\widetilde{\sigma}^{(j)}_{t,\kappa,n}}{\sigma^{(0)}_{t,\kappa,n}}\Delta^{n,2}_{t,i}W^{(0)}\Delta^{n,2}_{t,i}\widetilde{W}^{(j)}\right],$$

$$\chi^{(b)}_{i,t,\kappa,n} = 2\left(\frac{1}{N}\sum_{j=1}^N\left(\beta^{(j)}_{t,\kappa,n}-1\right)^2\right)\frac{n}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\Delta^n_{t,i-1}W^{(0)}\Delta^n_{t,i}W^{(0)},$$

$$\chi^{(c)}_{i,t,\kappa,n} = \frac{1}{N}\frac{n}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\sum_{j=1}^N\left[\left(\beta^{(j)}_{t,\kappa,n}-1\right)\frac{\gamma^{(j)}_{t,\kappa,n}}{\sigma^{(0)}_{t,\kappa,n}}\Delta^{n,2}_{t,i}W^{(0)}\Delta^{n,2}_{t,i}B\right],$$

and we note that we have

$$\frac{1}{N}\sum_{j=1}^N \overline{Z}^{(j)}_{t,\kappa,n} = \frac{1}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}\sqrt{N}}\widehat{Z}^{(a)}_{t,\kappa,n} + \frac{1}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\left(\widehat{Z}^{(b)}_{t,\kappa,n} + \widehat{Z}^{(c)}_{t,\kappa,n}\right).$$

LEMMA S10. *Assume Assumptions SA and B hold. For $n\to\infty$, $k_n\to\infty$ and $N\to\overline{N}$, with $\overline{N}\in(0,\infty]$, we have*

$$\left\{\widehat{Z}^{(a)}_{t,\kappa,n}, \widehat{Z}^{(b)}_{t,\kappa,n}, \widehat{Z}^{(c)}_{t,\kappa,n}\right\}_{t\in\mathcal{T},\kappa\in\mathcal{K}} \xrightarrow{\mathcal{L}-s} \left\{\sqrt{\psi^{(a)}_{t,\kappa}}Z^{(a)}_{t,\kappa}, \sqrt{\psi^{(b)}_{t,\kappa}}Z^{(b)}_{t,\kappa}, \sqrt{\psi^{(c)}_{t,\kappa}}Z^{(c)}_{t,\kappa}\right\}_{t\in\mathcal{T},\kappa\in\mathcal{K}},$$

for $t\in\mathcal{T}$, $\kappa\in\mathcal{K}$, with $\mathcal{K}$ being an arbitrary finite set of distinct points in $(0,1)$, and $\{Z^{(a)}_{t,\kappa}\}_{t\in\mathcal{T},\kappa\in\mathcal{K}}$, $\{Z^{(b)}_{t,\kappa}\}_{t\in\mathcal{T},\kappa\in\mathcal{K}}$ and $\{Z^{(c)}_{t,\kappa}\}_{t\in\mathcal{T},\kappa\in\mathcal{K}}$ being three sequences of i.i.d. standard normal random variables defined on an extension of the original probability space and independent of $\mathcal{F}$ and each other.

PROOF OF LEMMA S10. We denote

$$\widetilde{\chi}^{(k)}_{i,t,\kappa,n} = \chi^{(k)}_{i,t,\kappa,n} - \mathbb{E}^n_{t,i}\left(\chi^{(k)}_{i,t,\kappa,n}\right) + \mathbb{E}^n_{t,i+1}\left(\chi^{(k)}_{i+1,t,\kappa,n}\right) - \mathbb{E}^n_{t,i}\left(\chi^{(k)}_{i+1,t,\kappa,n}\right), \quad k = a,b,c.$$

Then, for $k = a, b, c$, we have

$$\mathbb{E}^n_{t,\lfloor \kappa n\rfloor - k_n}\Big(\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\big(\widetilde{\chi}^{(k)}_{i,t,\kappa,n} - \chi^{(k)}_{i,t,\kappa,n}\big)\Big) = 0, \qquad \mathbb{E}^n_{t,\lfloor \kappa n\rfloor - k_n}\Big(\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\big(\widetilde{\chi}^{(k)}_{i,t,\kappa,n} - \chi^{(k)}_{i,t,\kappa,n}\big)\Big)^2 \le \frac{K}{k_n},$$

and from here

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\big(\widetilde{\chi}^{(k)}_{i,t,\kappa,n} - \chi^{(k)}_{i,t,\kappa,n}\big) = o_p(1), \quad k = a, b, c.$$

Therefore, it suffices to prove the convergence result with $\chi^{(k)}_{i,t,\kappa,n}$ replaced by $\widetilde{\chi}^{(k)}_{i,t,\kappa,n}$ in $\widehat{Z}^{(k)}_{t,\kappa,n}$. Such a convergence result will follow by an application of Theorem IX.7.3 of Jacod and Shiryaev (2003) (by noting that $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}) = 0$) if we show the following convergence results:

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}\big(\widetilde{\chi}^{(a)}_{i,t,\kappa,n}\big)^2 \xrightarrow{\mathbb{P}} \psi^{(a)}_{t,\kappa}, \qquad \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}\big(\widetilde{\chi}^{(b)}_{i,t,\kappa,n}\big)^2 \xrightarrow{\mathbb{P}} \psi^{(b)}_{t,\kappa},$$
$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}\big(\widetilde{\chi}^{(c)}_{i,t,\kappa,n}\big)^2 \xrightarrow{\mathbb{P}} \psi^{(c)}_{t,\kappa}, \tag{S16}$$

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}\big(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\widetilde{\chi}^{(l)}_{i,t,\kappa,n}\big) \xrightarrow{\mathbb{P}} 0, \quad k, l = a, b, c \text{ with } k \neq l, \tag{S17}$$

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}\big(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M\big) \xrightarrow{\mathbb{P}} 0, \quad k = a, b, c, \tag{S18}$$

for M being $W^{(0)}$, a component of B, or a bounded martingale orthogonal to them (in a martingale sense). The first two convergence results in (S16) and (S17) follow directly by taking into account that the volatility processes and the beta process all have càdlàg paths. The last convergence result in (S18) when M is $W^{(0)}$ or a component of B holds trivially because due to the symmetry of the standard normal distribution, $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) = 0$ in this case.

Suppose now that M in (S18) is equal to a bounded martingale orthogonal to $W^{(0)}$ and B. First, $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) = 0$ for $k = b, c$ because M is orthogonal to $W^{(0)}$ and B. Second, if M is a discontinuous martingale, we again trivially have $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) = 0$ for $k = a, b, c$. Thus, we are left with showing (S18) with $k = a$ and M being a continuous bounded martingale that is orthogonal to $W^{(0)}$ and B. In this case, we can write

$$\mathbb{E}^n_{t,i}\big(\Delta^n_{t,i-1}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}\Delta^n_{t,i}M\big) = 0,$$

and

$$\mathbb{E}^n_{t,i}\big(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}\Delta^n_{t,i}M\big) = \mathbb{E}^n_{t,i}\big(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}Z^N_{t-1+i/n}\big),$$

where $Z^N_s = \mathbb{E}(\Delta^n_{t,i}M|\widetilde{\mathcal{F}}^{(N)}_s)$ for $s \in [t-1+(i-1)/n, t-1+i/n]$ and for $\{\widetilde{\mathcal{F}}^{(N)}_t\}_{t\ge 0}$ being the filtration generated by the Brownian motions $W^{(0)}$, B and $\{\widetilde{W}^{(j)}\}_{j=1,\ldots,N}$. Note that

Z_s^N is a $\widetilde{\mathcal{F}}^{(N)}$-martingale for $s \in [t-1+(i-1)/n, t-1+i/n]$. Therefore, by a martingale representation theorem, the orthogonality of M to $W^{(0)}$ and B, the fact that $W^{(0)}$ and B are independent from each other, and using integration by parts, we have

$$\begin{aligned}
&\mathbb{E}_{t,i}^n\big(\Delta_{t,i}^n W^{(0)} \Delta_{t,i}^n \widetilde{W}^{(j)} Z_{t-1+i/n}^N\big) \\
&\quad = \mathbb{E}_{t,i}^n\left(\Delta_{t,i}^n W^{(0)} \Delta_{t,i}^n \widetilde{W}^{(j)} \sum_{j=1}^N \int_{t-1+(i-1)/n}^{t-1+i/n} \eta_s^{(j)}\, d\widetilde{W}_s^{(j)}\right) \\
&\quad = \mathbb{E}_{t,i}^n\left(\sum_{j=1}^N \int_{t-1+(i-1)/n}^{t-1+i/n} \big(W_s^{(0)} - W_{t-1+(i-1)/n}^{(0)}\big)\eta_s^{(j)}\, ds\right),
\end{aligned}$$

where $\{\eta_s^{(j)}\}_{j=1,\dots,N}$ are some $\mathcal{F}_s$-adapted processes. From here, using the shorthand notation $\zeta_{t,\kappa,n}^{(j)} = (\beta_{t,\kappa,n}^{(j)} - 1)\widetilde{\sigma}_{t,\kappa,n}^{(j)}$, we can write

$$\begin{aligned}
&\sum_{j=1}^N \left(\zeta_{t,\kappa,n}^{(j)} \sum_{i\in\widetilde{\mathcal{I}}_\kappa^n} \mathbb{E}_{t,i}^n\big(\Delta_{t,i}^n W^{(0)} \Delta_{t,i}^n \widetilde{W}^{(j)} \Delta_{t,i}^n M\big)\right) \\
&\quad = \sum_{i\in\widetilde{\mathcal{I}}_\kappa^n} \mathbb{E}_{t,i}^n\left(\sum_{j=1}^N \zeta_{t,\kappa,n}^{(j)} \int_{t-1+(i-1)/n}^{t-1+i/n} \big(W_s^{(0)} - W_{t-1+(i-1)/n}^{(0)}\big)\eta_s^{(j)}\, ds\right).
\end{aligned}$$

From here, by applying Cauchy–Schwarz inequality, we have

$$\left|\mathbb{E}_{t,i}^n\left(\int_{t-1+(i-1)/n}^{t-1+i/n} \big(W_s^{(0)} - W_{t-1+(i-1)/n}^{(0)}\big)\eta_s^{(j)}\, ds\right)\right| \le \frac{K}{\sqrt{n}} \int_{t-1+(i-1)/n}^{t-1+i/n} \sqrt{\mathbb{E}_{t,i}^n\big(\eta_s^{(j)}\big)^2}\, ds.$$

Using inequality in means, we have

$$\sum_{j=1}^N \sqrt{\mathbb{E}_{t,i}^n\big(\eta_s^{(j)}\big)^2} \le K\sqrt{N}\sqrt{\mathbb{E}_{t,i}^n\left(\sum_{j=1}^N \big(\eta_s^{(j)}\big)^2\right)},$$

and therefore by another application of equality in means, we have

$$\begin{aligned}
\sum_{j=1}^N \int_{t-1+(i-1)/n}^{t-1+i/n} \sqrt{\mathbb{E}_{t,i}^n\big(\eta_s^{(j)}\big)^2}\, ds &\le K\sqrt{\frac{N}{n}}\sqrt{\mathbb{E}_{t,i}^n\left(\int_{t-1+(i-1)/n}^{t-1+i/n} \sum_{j=1}^N \big(\eta_s^{(j)}\big)^2\, ds\right)} \\
&\le K\sqrt{\frac{N}{n}}\sqrt{\mathbb{E}_{t,i}^n\left(\int_{t-1+(i-1)/n}^{t-1+i/n} d\langle Z^N, Z^N\rangle_s\right)}.
\end{aligned}$$

Applying again inequality in means, we can finally write

$$\mathbb{E}\left(\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n} \sum_{j=1}^N \int_{t-1+(i-1)/n}^{t-1+i/n} \sqrt{\mathbb{E}_{t,i}^n\big(\eta_s^{(j)}\big)^2}\, ds\right) \le K\sqrt{k_n}\sqrt{\frac{N}{n}}\sqrt{\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n} \mathbb{E}\big(\Delta_{t,i}^n\langle Z^N, Z^N\rangle\big)}.$$

Now, using the definition of the martingale Z^N, successive conditioning and Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\big(\Delta_{t,i}^n\langle Z^N, Z^N\rangle\big) &= \mathbb{E}\big(\mathbb{E}\big(\Delta_{t,i}^n M|\widetilde{\mathcal{F}}_{t-1+i/n}^{(N)}\big)\big)^2 - \mathbb{E}\big(\mathbb{E}\big(\Delta_{t,i}^n M|\widetilde{\mathcal{F}}_{t-1+(i-1)/n}^{(N)}\big)\big)^2 \\ &\leq 2\mathbb{E}\big(\Delta_{t,i}^n M\big)^2.\end{aligned}$$

Because of the boundedness of the martingale M and its continuity, we therefore have

$$\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n} \mathbb{E}\big(\Delta_{t,i}^n\langle Z^N, Z^N\rangle\big) \leq 2\mathbb{E}\big(\langle M, M\rangle_{t-1+\frac{\lfloor\kappa n\rfloor}{n}} - \langle M, M\rangle_{t-1+\frac{\lfloor\kappa n\rfloor - k_n+1}{n}}\big) \downarrow 0,$$

as $k_n/n \to 0$. □

Combining Lemmas S3, S5–S8, we have for $\varpi \in (1/4, 1/2)$, $\varrho \in (0, 1/2)$ with $\varrho > 2 - 4\varpi$:

$$\widehat{D}_{t,\kappa} - \widehat{B}_{t,\kappa} - D_{t,\kappa}^N = \frac{1}{N}\sum_{j=1}^N \overline{Z}_{t,\kappa,n}^{(j)} + O_p\left(\frac{1}{\alpha_n^4}\left(\frac{1}{k_n} \vee \sqrt{\frac{k_n}{n}} \vee \Delta_n^{\varpi}\right)\right).$$

Moreover, from Lemma S9 and Assumption B, under the same conditions for ϖ and ϱ as above,

$$\frac{\widehat{\mathrm{Avar}}\big(\widehat{D}_{t,\kappa}^N\big)}{\mathrm{Avar}\big(\widehat{D}_{t,\kappa}^N\big)} \xrightarrow{\mathbb{P}} 1, \qquad \frac{\big|\widetilde{\mathcal{I}}_\kappa^n\big|\,\mathrm{Avar}\big(\widehat{D}_{t,\kappa}^N\big)}{\psi_{t,\kappa}^{(a)}/N + \psi_{t,\kappa}^{(b)} + \psi_{t,\kappa}^{(c)}} \xrightarrow{\mathbb{P}} 1.$$

Combining these results with Lemma S10, we get the result of the theorem.

S1.6 *Proof of Theorem 2*

We start with showing the counterpart of Lemma S10 in the current context. The result of Lemma S11 below is slightly more restrictive than what we showed in Lemma S10 when $\overline{N} = \infty$. Nevertheless, it suffices for the purposes of proving Theorem 2.

LEMMA S11. *Assume Assumptions SA and B hold and let $\{\omega_l\}_{l\in I}$ be Lipschitz real-valued continuous functions on $[0, 1]$, where I is a countable set. For $n \to \infty$, $k_n \to \infty$ and $N \to \overline{N}$, with $\overline{N} \in (0, \infty)$, we have*

$$\sqrt{\frac{k_n}{n}}\left\{\begin{array}{l}\displaystyle\sum_{s=k_n}^n \omega_l(s\Delta_n)\widehat{Z}_{t,s\Delta_n,n}^{(a)} \\ \displaystyle\sum_{s=1}^n \omega_l(s\Delta_n)\widehat{Z}_{t,s\Delta_n,n}^{(b)} \\ \displaystyle\sum_{s=k_n}^n \omega_l(s\Delta_n)\widehat{Z}_{t,s\Delta_n,n}^{(c)}\end{array}\right._{t\in\mathcal{T}, i\in I}$$

$$\xrightarrow{\mathcal{L}-s} \left\{ \begin{array}{l} \int_{t-1}^{t} \omega_i\big(s-\lfloor s\rfloor\big)\sqrt{\psi^{(a)}_{t,s-\lfloor s\rfloor}}\, dZ^{(a)}_s \\ \int_{t-1}^{t} \omega_i\big(s-\lfloor s\rfloor\big)\sqrt{\psi^{(b)}_{t,s-\lfloor s\rfloor}}\, dZ^{(b)}_s \\ \int_{t-1}^{t} \omega_i\big(s-\lfloor s\rfloor\big)\sqrt{\psi^{(c)}_{t,s-\lfloor s\rfloor}}\, dZ^{(c)}_s \end{array} \right.\Bigg._{t\in\mathcal{T},i\in I}, \tag{S19}$$

where $Z^{(a)}_s$, $Z^{(b)}_s$ and $Z^{(c)}_s$ are three independent Brownian motions sequences defined on an extension of the original probability space and independent of $\mathcal{F}$. If in the above setting $\overline{N}=\infty$, then the convergence result in (S19) for the sums involving $\widehat{Z}^{(b)}_{t,s\Delta_n,n}$ and $\widehat{Z}^{(c)}_{t,s\Delta_n,n}$ continues to hold.

Proof of Lemma S11. Using the notation and the bounds derived in Lemma S10, we have

$$\sqrt{\frac{k_n}{n}}\sum_{s=k_n}^{n}\sum_{i\in\widetilde{\mathcal{I}}^n_{s\Delta_n}} \omega_l(s\Delta_n)\big(\widetilde{\chi}^{(k)}_{t,i,s\Delta_n,n}-\chi^{(k)}_{t,i,s\Delta_n,n}\big)=O_p(1/\sqrt{k_n}), \quad k=a,b,c.$$

From here, the proof of the lemma follows exactly the same steps as corresponding ones in the proof of Lemma S10. □

Combining Lemmas S3, S5–S8, we have for $\varpi\in(1/4,1/2)$, $\varrho\in(0,1/2)$ with $\varrho>2-4\varpi$ and for ω as in Lemma S11 above:

$$\begin{aligned} &\frac{1}{n-k_n+1}\sum_{s=k_n}^{n}\big[\omega(s\Delta_n)\big(\widehat{D}^N_{t,s\Delta_n}-\widehat{B}^N_{t,s\Delta_n}\big)\big]-\int_{t-1}^{t}\omega(s-t+1)D^N_{t,s-\lfloor s\rfloor}\,ds \\ &=\frac{1}{n-k_n+1}\sum_{s=k_n}^{n}\omega(s\Delta_n)\frac{1}{N}\sum_{j=1}^{N}\overline{Z}^{(j)}_{t,s\Delta_n,n}+O_p\left(\frac{1}{\alpha_n^4}\left(\frac{1}{k_n^{3/2}}\vee\left(\frac{k_n}{n}\right)^{1-\iota}\vee\Delta_n^{2\varpi}\right)\right). \end{aligned}$$

Moreover, from Lemma S9 and Assumption B, under the same conditions for ϖ and ϱ as above, we have for arbitrary ω, ω' that are Lipschitz real-valued continuous functions on $[0,1]$:

$$\frac{\sum_{s=1}^{n}\big[\overline{\omega}^n(s)\overline{\omega}'^n(s)\big(\widehat{\mathrm{Avar}}\big(\widehat{D}^N_{t,(s\vee k_n)\Delta_n}\big)+\widehat{\mathrm{Avar}}\big(\widehat{D}^N_{t,(s+k_n\wedge n)\Delta_n}\big)\big)\big]}{\sum_{s=1}^{n}\big[\overline{\omega}^n(s)\overline{\omega}'^n(s)\big(\mathrm{Avar}\big(\widehat{D}^N_{t,(s\vee k_n)\Delta_n}\big)+\mathrm{Avar}\big(\widehat{D}^N_{t,(s+k_n\wedge n)\Delta_n}\big)\big)\big]}\xrightarrow{\mathbb{P}}1,$$

and

$$\frac{1}{2}\frac{k_n}{n}\frac{\sum_{s=1}^{n}\big[\overline{\omega}^n(s)\overline{\omega}'^n(s)\big(\mathrm{Avar}\big(\widehat{D}^N_{t,(s\vee k_n)\Delta_n}\big)+\mathrm{Avar}\big(\widehat{D}^N_{t,(s+k_n\wedge n)\Delta_n}\big)\big)\big]}{\int_{t-1}^{t}\big[\omega(s-t+1)\omega'(s-t+1)\big(\psi^{(a)}_{t,s-\lfloor s\rfloor}/N+\psi^{(b)}_{t,s-\lfloor s\rfloor}+\psi^{(c)}_{t,s-\lfloor s\rfloor}\big)\big]\,ds}\xrightarrow{\mathbb{P}}1,$$

where we define $\overline{\omega}^n(i)$ and $\overline{\omega}'^n(i)$, for $i = 1, \ldots, n$, from ω and ω' exactly as in equation (7) in the paper.

Furthermore, the above two convergences hold uniformly for ω, ω' belonging to the set of weighting functions of the theorem. Overall, the above results, together with Lemma S11 imply the convergence result of the theorem holds finite-dimensionally, that is, for any finite set of points in $\mathcal{U}$. Therefore, we are left with showing tightness of the sequence in the space of continuous functions on $\mathcal{U}$ equipped with the uniform topology. For this, we make use of Theorem 12.3 of Billingsley (2013) and Lemmas S3, S5–S8 as well as the smoothness of $\omega_u(z)$ in u assumed in the statement of the theorem.

S1.7 *Proof of Corollary 1*

Part (a) follows from Theorem 2 while part (b) follows from Theorem 1 of Bierens (1982).

S1.8 *Proof of Theorem 3*

Throughout this proof, $\|f\| = \langle f, f\rangle$ is the norm of $f \in \mathcal{L}^2(w)$. We start the proof with denoting the function

$$g(u, x, y) = e^{iu\frac{y}{x} + iu}, \quad \text{for } u, x, y \in \mathbb{R}.$$

Using Taylor expansion, we then have the decomposition

$$g\big(u, \widehat{V}_{t,\kappa}^{(0,j)}, \widehat{C}_{t,\kappa}^{(j)}(2)\big) - g\big(u, \widetilde{V}_{t,\kappa}^{(0)}, \widetilde{C}_{t,\kappa}^{(j)}\big) = Z_{t,\kappa,n}^{(j)}(u) + R_{t,\kappa}^{(j)}(u),$$

where

$$Z_{t,\kappa,n}^{(j)}(u) = \nabla_x g\big(u, \widetilde{V}_{t,\kappa}^{(0)}, \widetilde{C}_{t,\kappa}^{(j)}\big)\big(\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}\big) + \nabla_y g\big(u, \widetilde{V}_{t,\kappa}^{(0)}, \widetilde{C}_{t,\kappa}^{(j)}\big)\big(\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}\big),$$

and the residual term $R_{t,\kappa}^{(j)}(u)$ satisfies

$$\big|R_{t,\kappa}^{(j)}(u)\big| \le K\big(|u| \vee 1\big)1_{\{|\widehat{V}_{t,\kappa}^{(0,j)}| \le \alpha_n\}} + K\big(|u|^2 \vee 1\big)\Big(\big(\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}\big)^2 + \big(\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}\big)^2\Big),$$

with a constant K that does not depend on n, j and u. We further denote

$$\begin{aligned}
&\overline{Z}_{t,\kappa,n}^{(j)}(u) \\
&\quad = e^{iu\beta_{t,\kappa,n}^{(j)}} iu\big(\beta_{t,\kappa,n}^{(j)} - 1\big)^2 \frac{n}{\big|\widetilde{\mathcal{I}}_\kappa^n\big|} \sum_{i \in \widetilde{\mathcal{I}}_\kappa^n} \Delta_{t,i-1}^n W^{(0)} \Delta_{t,i}^n W^{(0)} \\
&\qquad + \frac{1}{2} e^{iu\beta_{t,\kappa,n}^{(j)}} iu\big(\beta_{t,\kappa,n}^{(j)} - 1\big)\Bigg(\frac{n}{\big|\widetilde{\mathcal{I}}_\kappa^n\big|} \sum_{i \in \widetilde{\mathcal{I}}_\kappa^n} \frac{\gamma_{t,\kappa,n}^{(j)}}{\sigma_{t,\kappa,n}^{(0)}} \Delta_{t,i}^{n,2} W^{(0)} \Delta_{t,i}^{n,2} B \\
&\qquad + \frac{n}{\big|\widetilde{\mathcal{I}}_\kappa^n\big|} \sum_{i \in \widetilde{\mathcal{I}}_\kappa^n} \frac{\widetilde{\sigma}_{t,\kappa,n}^{(0)}}{\sigma_{t,\kappa,n}^{(0)}} \Delta_{t,i}^{n,2} W^{(0)} \Delta_{t,i}^{n,2} \widetilde{W}^{(j)}\Bigg).
\end{aligned}$$

Using Lemmas S1–S3 and Lemma S8 and the exponential tail decay of the weighting function w, we have

$$\mathbb{E}\left\|\frac{1}{N}\sum_{j=1}^{N}R_{t,\kappa}^{(j)}\right\| \leq \frac{K}{\alpha_n^2}\left(\frac{1}{k_n}\vee\frac{k_n}{n}\right),$$

where we have made use of the fact that $\varpi > 1/4$. Similarly, using Lemmas S1–S3, we have that

$$\mathbb{E}\left\|\frac{1}{N}\sum_{j=1}^{N}(Z_{t,\kappa,n}^{(j)}-\overline{Z}_{t,\kappa,n}^{(j)})\right\| \leq \frac{K}{\alpha_n^2}\left(\frac{1}{k_n}\vee\frac{k_n}{n}\right) \leq K\left(\Delta_n^{2\varpi}\vee\sqrt{\frac{k_n}{n}}\vee\frac{1}{k_n}\right).$$

Given the rate condition on the sequence k_n, we are left with showing

$$\sqrt{k_n}\frac{1}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)} \xrightarrow{\mathcal{L}-s} \mathcal{Z}_{t,\kappa},$$

with $\mathcal{Z}_{t,\kappa}$ being the limit in the statement of the theorem. Using Bessel's inequality and dominated convergence, we have

$$\mathbb{E}\left(\sum_{i>I}\left\langle\frac{k_n}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)}, e_i\right\rangle^2\right) \to 0, \quad \text{as } I\to\infty,$$

where $\{e_i\}_{i\geq 1}$ denotes an orthonormal basis in $\mathcal{L}^2(w)$. This means that the sequence is asymptotically finite-dimensional; see 1.8 in Vaart and Wellner (1996). Therefore, the limit result of the theorem will follow from Theorem 1.8.4 in Vaart and Wellner (1996) if we can establish

$$\left\langle\frac{\sqrt{k_n}}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)}, h\right\rangle \xrightarrow{\mathcal{L}-s} \langle\mathcal{Z}_{t,\kappa}, h\rangle,$$

for $\mathcal{Z}_{t,\kappa}$ denoting the limit of Theorem 1 and h an arbitrary element in $\mathcal{L}^2(w)$. This convergence follows by an application of Lemma S11.

Appendix S2: Additional evidence

S2.1 *Cross-sectional dispersion of betas and jumps*

In the paper, we eliminate all jumps in the individual assets as well as the market. To verify that the documented contraction of betas over the trading day is not due to this truncation procedure, we reproduce the quantile plot of Figure 3 in the paper using the following standard estimator of beta, which includes returns with jumps:

$$\widetilde{\beta}_{t,\kappa}^{(j)} = \frac{\sum_{i\in\mathcal{I}_\kappa^n}\Delta_{t,i}^n X^{(j)}\Delta_{t,i}^n X^{(0)}}{\sum_{i\in\mathcal{I}_\kappa^n}\left(\Delta_{t,i}^n X^{(0)}\right)^2}, \quad j=1,\ldots,N. \tag{S20}$$

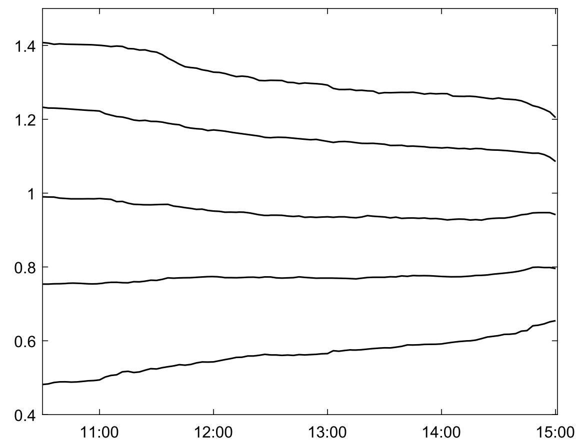

FIGURE S1. Cross-sectional distribution of total market betas across the trading day. The figure plots the cross-sectional quantiles of the un-truncated betas defined in equation (S20). All quantities are treated as functions of the trading day and computed by averaging over the entire sample. The selected quantiles are: 10th, 25th, 50th, 75th, and 90th.

The results are displayed in Figure S1. As seen from the figure, the impact of truncation on the cross-sectional distribution of betas is minimal. Furthermore, the strong contraction of betas towards unity is preserved even when jumps are included.

S2.2 *Cross-sectional dispersion of betas and microstructure effects*

While stocks can be traded continuously throughout the trading day, only a finite number of trades of a given stock are made over any fixed period of time. These trade times are not necessarily the same across stocks. This asynchronicity will cause the estimated covariances to be downward biased (Epps effect). Note that the sign of this bias is independent of the stocks beta, thus if staleness is to explain our findings it would need to be high (low) in the morning and low (high) near the close for low (high) beta stocks.

To assess whether this is the case, we compute for each stock the average staleness at any time of the day. In the left panel of Figure S2 we plot the cross-sectional average staleness over the trading day. Several things are worth highlighting. First, staleness is below 12 seconds across the trading day, which is low when compared to the 6-minute frequency used in calculating the covariances. Thus, staleness is not going to have a major effect on our inference procedures. Second, staleness gradually increases during the first part of the trading day, and then decreases over the second-half of the trading day, ending at an average staleness of less than 1 second.

However, this does not address the question of whether staleness of high and low beta stocks evolves differently. To access whether this is the case, we sort the stocks based on their beta at the open into 5 groups. For each group of stocks, we compute the average staleness. The resulting series are plotted in the right panel of Figure S2. The key takeaway is that high and low beta stocks exhibit the same intraday pattern in staleness, thus staleness cannot explain our findings.

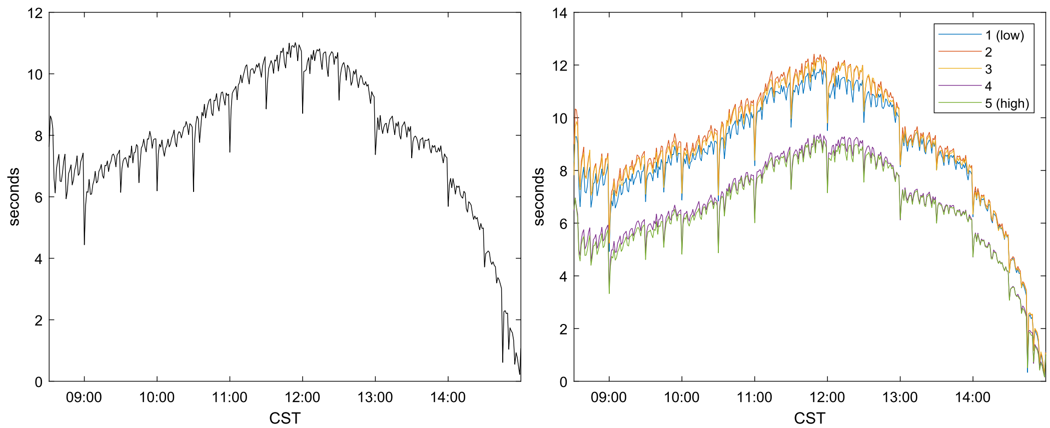

FIGURE S2. Staleness of prices. The left figure plots the cross-sectionally averaged average staleness. The right figure plots the average staleness for of prices of stocks sorted according to their beta at market open.

As a secondary robustness check for potential adverse effects on our results from the presence of market microstructure noise, we compute the cross-sectional dispersion of market betas using 30 minute returns (thus relying on the time-series dimension rather than infill one). We then compute the cross-sectional deviation from 1, that is, $\frac{1}{N}\sum_{j=1}^{N}(\bar{\beta}_{j,k}-1)^2$, where $\bar{\beta}_{j,k}$ is the OLS estimate of market beta in the following time-series regression:

$$r_{t,k}^{(j)} = \alpha_j + \beta_j r_{t,k}^{(0)} + \epsilon_{t,k}^{(j)}, \quad t = 1, \ldots, T, \tag{S21}$$

where $r_{t,k}^{(j)}$ is the return of the asset j over the k'th, $k = 1, \ldots, 13$, half hour interval of the trading day on day t. The result is plotted in Figure S3.

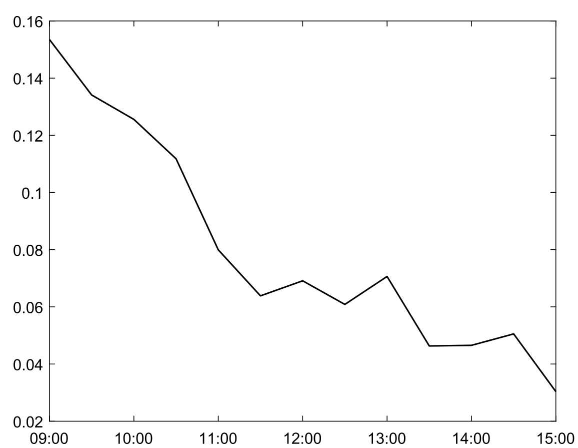

FIGURE S3. Cross-sectional dispersion of low-frequency betas. The figure plots the cross-sectional dispersion of betas estimated using 30 minute returns according to equation (S21).

Note that this approach differs in one important way from the one taken in the paper. Here, we are computing the dispersion of time-series averaged betas, whereas in the main text we compute the time-series average of the cross-sectional dispersion of betas. The dispersion presented below will be lower by construction. However, the finding of monotonically decreasing cross-sectional dispersion of market betas during the day can be clearly seen even when using lower frequency returns.

References

Bierens, H. J. (1982), "Consistent model specification tests." *Journal of Econometrics*, 20 (1), 105–134. [18]

Billingsley, P. (2013), *Convergence of Probability Measures*. John Wiley & Sons. [18]

Jacod, J. and P. Protter (2012), *Discretization of Processes*. Springer-Verlag, Berlin. [5]

Jacod, J. and A. Shiryaev (2003), *Limit Theorems for Stochastic Processes*, second edition. Springer-Verlag, Berlin. [14]

Vaart, A. W. and J. Wellner (1996), *Weak Convergence and Empirical Processes*. Springer, New York, NY. [19]

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